Some stuff in this talk has been explained on the blackboard, e.g.:

Def.: Ordered *-algebra

An ordered *-algebra is an associative $\mathbb{C}$-algebra $\mathcal{A}$ with unit $\mathbb{1} \in \mathcal{A}$, antilinear *-involution $\cdot^* \colon \mathcal{A} \to \mathcal{A}$ fulfilling $(ab)^* = b^* a^*$ and $(a^*)^* = a$ for all $a, b \in \mathcal{A}$, and a partial order $\leq$ on the real linear subspace $\mathcal{A}_H := \{ a \in \mathcal{A} \mid a^* = a \}$ of Hermitian elements in $\mathcal{A}$, such that

$$a + c \leq b + c \,, \qquad d^* a\, d \leq d^* b\, d \qquad \text{and} \qquad 0 \leq \mathbb{1}$$

hold for all $a, b, c \in \mathcal{A}_H$ with $a \leq b$ and all $d \in \mathcal{A}$.

Exercise

Where in this talk should one drop words like “Hamilton’s equations” or “Schrödinger equation”?

Introduction to Deformation Quantization

Matthias Schötz

ULB

06.02.2020

- Classical physics (since Galileo / Newton, ...):
  - Describes our world on “not too small” scales.
  - E.g.: classical mechanics describes the behaviour of finitely many pointlike particles (solar system, canon balls, pendulum, ...)
  - Mathematical description by manifolds, fibre bundles, differential equations, ...
- Quantum physics (since 1900, Planck, ...):
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  - Depends on Planck's constant $\hbar \approx 10^{-34}\ kg\ m^2\ s^{-1}$.
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  - Classical and quantum physics look and behave very differently.
  - Classical physics is intuitive and “easy”, quantum physics is not.
  - But classical physics should be an approximation to quantum physics.

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Idea

$$\text{classical physics} = \lim_{\hbar \to 0} \text{quantum physics}$$

Outline

- Harmonic Oscillator in Classical Mechanics and Quantum Mechanics
- Wick Star Product
- Generalizations

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Harmonic Oscillator in Classical Mechanics

- Phase space: $\mathbb{R}^2$
- Observables: Real-valued polynomial functions in $q := \mathrm{pr}_1 : \mathbb{R}^2 \to \mathbb{R}$ and $p := \mathrm{pr}_2 : \mathbb{R}^2 \to \mathbb{R}$.
- Hamiltonian: $H := \frac{p^2+q^2}{2}$.
- Equations of motion: A smooth curve $\gamma \colon \mathbb{R} \to \mathbb{R}^2$ is a solution of the equations of motion iff

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\tau} q(\gamma(t)) = \frac{\partial H}{\partial p}\bigg|_{\gamma(\tau)} = p(\gamma(\tau))$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\tau} p(\gamma(t)) = -\frac{\partial H}{\partial q}\bigg|_{\gamma(\tau)} = -q(\gamma(\tau))$$

hold for all $\tau \in \mathbb{R}$.

Harmonic Oscillator in Quantum Mechanics

- Phase space: $\left\{ \psi \in \mathcal{S}(\mathbb{R})/U(1) \;\middle|\; ||\psi|| = 1 \right\}$, where $\mathcal{S}$ denotes the pre-Hilbert space of rapidly decreasing smooth complex-valued functions.
- Observables: Hermitian (complex) polynomials in $q, p \in \mathcal{L}^*\big(\mathcal{S}(\mathbb{R})\big)$, where

$$\big(q\psi\big)(x) := x\psi(x) \qquad \text{and} \qquad p\psi := -\mathrm{i}\hbar\psi'.$$

- Hamiltonian: $H := \frac{p^2+q^2}{2}$.
- Equations of motion: A smooth curve $\gamma\colon \mathbb{R} \to \left\{ \psi \in \mathcal{S}(\mathbb{R})/U(1) \;\middle|\; ||\psi|| = 1 \right\}$ is a solution of the equations of motion iff

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\tau} \gamma(t) = H\gamma(\tau) = \frac{1}{2}\Big(-\hbar^2\gamma(\tau)'' + q^2\big(\gamma(\tau)\big)\Big)$$

holds for all $\tau \in \mathbb{R}$.

Comparison of classical and quantum description

Similarities

- Observables are polynomials in position-observable q and momentum-observable p.
  These are the Hermitian elements of an ordered *-algebra.
- Hamiltonian is $H = \frac{p^2+q^2}{2}$.

Differences

- Phase spaces seem to be different.
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States of ordered *-algebras:

Let $\mathcal{A}$ be an ordered *-algebra, then a *state* on $\mathcal{A}$ is a linear functional $\phi \colon \mathcal{A} \to \mathbb{C}$ with the following properties:

- ϕ is Hermitian, i.e. $\langle \phi , a^* \rangle = \overline{\langle \phi , a \rangle}$ for all $a \in \mathcal{A}$.
- ϕ is positive, i.e. $\langle \phi , a \rangle \geq 0$ for all positive Hermitian $a \in \mathcal{A}$.
- ϕ is normalized, i.e. $\langle \phi , \mathbb{1} \rangle = 1$.
- If $\mathcal{A}$ is the classical observable algebra of polynomial functions on $\mathbb{R}^2$, then the points $x \in \mathbb{R}^2$ give states $\delta_x \colon \mathcal{A} \to \mathbb{C}$,
$$f \mapsto \langle \delta_x , f \rangle := f(x).$$
- If $\mathcal{A}$ is the quantum observable algebra of operators on $\mathcal{S}(\mathbb{R})$, then the normalized vectors $\psi \in \mathcal{S}(\mathbb{R})/U(1)$ give states $\omega_\psi \colon \mathcal{A} \to \mathbb{C}$,
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Equations of motion

A smooth curve γ from $\mathbb{R}$ to the states of the observable algebra $\mathcal{A}$ fulfils the equations of motion iff

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\tau} \langle\, \gamma(t)\,,\, a \,\rangle = \langle\, \gamma(\tau)\,,\, \{\, a\,,\, H\,\}\,\rangle$$

holds for all $a \in \mathcal{A}$.

- In classical mechanics, $\{\,\cdot\,,\,\cdot\,\}$ is the standard Poisson bracket
$$\{\, a\,,\, b\,\} := \frac{\partial a}{\partial q}\frac{\partial b}{\partial p} - \frac{\partial b}{\partial q}\frac{\partial a}{\partial p}$$
for smooth functions a, b on $\mathbb{R}^2$.
- In quantum mechanics, $\{\,\cdot\,,\,\cdot\,\}$ is the commutator
$$\{\, a\,,\, b\,\} := \frac{ab - ba}{\mathrm{i}\hbar}$$
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Unified description of classical and quantum mechanics

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Idea of Deformation Quantization

- Describe the observable algebra of classical and quantum physics by the same vector space.
- The vector space of the classical observable algebra is a good choice: We understand that quite well, just functions on the “intuitive” classical phase space.
- The product of the observable algebra depends on $\hbar$, this is the star product $\star_\hbar$.
- In the limit $\hbar \to 0$, we should get the classical, pointwise product $\star_0$.
- The bracket $\{ \cdot\, , \, \cdot \}_\hbar$ also depends on $\hbar$. We have

$$\{ a\, , \, b \}_\hbar = \frac{a \star_\hbar b - b \star_\hbar a}{\mathrm{i}\hbar}$$

for $\hbar > 0$, and $\lim_{\hbar \to 0} \{ a\, , \, b \}_\hbar$ should be the classical Poisson bracket.

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Example: the Wick star product

For two polynomial functions a, b on $\mathbb{R}^2 \cong \mathbb{C}$ we define

$$a \star_\hbar b := \sum_{r=0}^{\infty} \frac{(2\hbar)^r}{r!} \frac{\partial^r a}{\partial z^r} \frac{\partial^r b}{\partial \overline{z}^r}$$

with

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial q} + \frac{1}{\mathrm{i}} \frac{\partial}{\partial p}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial q} - \frac{1}{\mathrm{i}} \frac{\partial}{\partial p}\right).$$

Everything up to now has been physics, so let's do math:

Let's generalize stuff!

This leads to:

- Formal deformation quantization on Poisson manifolds.
- Strict deformation quantization of C^*-algebras.
- Others...