New Goodness-of-Fit tests for the Weibull distribution

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18th October 2022

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Goodness-of-Fit test

Consider an i.i.d. sample $X_1, \dots, X_n \sim \mathbb{P}^X$. We are interested if the underlying distribution $\mathbb{P}^X$ equals a given probability distribution $\mathbb{Q}$. In terms of statistical testing

$$\mathcal{H}_0 : \mathbb{P}^X = \mathbb{Q} \quad \text{and} \quad \mathcal{H}_1 : \mathbb{P}^X \neq \mathbb{Q}.$$

An asymptotically consistent level-α test satisfies

$$\lim_{n\to\infty} \mathbb{P}(\neg\mathcal{H}_0) = \alpha \quad \text{under } \mathcal{H}_0 \quad \text{and} \quad \lim_{n\to\infty} \mathbb{P}(\neg\mathcal{H}_0) = 1 \quad \text{under } \mathcal{H}_1.$$

Example. With $\hat{F}_n(t) = \frac{1}{n}\sum_{i=1}^n 1_{\{X_i \le t\}}$ we define

$$KS = \sqrt{n} \sup_t |\hat{F}_n(t) - F_{\mathbb{Q}}(t)| \quad \text{(Kolmogorov-Smirnov)}.$$

It is well known that

$$KS \xrightarrow{D} \sup_{t\in[0,1]} |B(t)| \quad \text{(Kolmogorov distribution)},$$

where $B(\cdot)$ denotes the Brownian Bridge.

We obtain an asymptotically consistent level-α test if we reject $\mathcal{H}_0$ when $KS > c^*$, where c^* denotes the $(1-\alpha)$-quantile of the Kolmogorov distribution.

The Weibull distribution

The Weibull distribution $W(\lambda, k)$, $\lambda, k > 0$ is a probabilty distribution on the positive real numbers with pdf

$$f(x) = \frac{k}{\lambda^k} x^{k-1} \exp\left(-\left(\frac{x}{\lambda}\right)^k\right), \qquad x > 0.$$

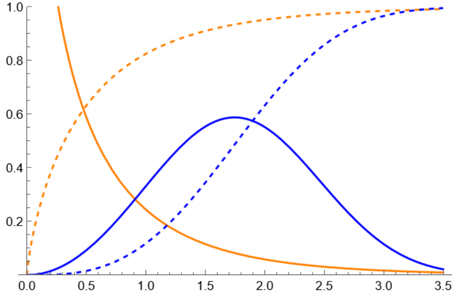

Figure: pdf and cdf for parameter values $(\lambda, k) = (0.5, 0.8)$ (orange) and $(\lambda, k) = (2, 3)$ (blue).

Given $X_1, \ldots, X_n \sim \mathbb{P}^X$ and $\mathcal{W} = \{W(\lambda, k) \mid \lambda, k > 0\}$ our test problem becomes

$$\mathcal{H}_0 : \mathbb{P}^X \in \mathcal{W} \qquad \text{and} \qquad \mathcal{H}_1 : \mathbb{P}^X \notin \mathcal{W}.$$

Stein characterizations

The density approach within the framework of Stein's states that a random variable X follows a probability distribution with density f if and only if

$$\mathbb{E}\left[p'(X) + \frac{f'(X)}{f(X)}p(X)\right] = 0$$

for all p belonging to a sufficiently large function class.

This characterization can be untied from the class of test functions: Under weak assumptions, a positive random variable X with density p_X follows a probability distribution with density f if and only if

$$p_X(t) = \mathbb{E}\left[-\frac{f'(X)}{f(X)}1_{\{X>t\}}\right], \qquad t > 0.$$

In the Weibull case we obtain

$$f(t) = \mathbb{E}\left[-\frac{k-1-kX^k/\lambda^k}{X}1_{\{X>t\}}\right], \qquad t > 0.$$

Stein characterizations

With the latter density representation, we get a new characterization of the Weibull distribution based on the LaPlace transform $\mathcal{L}_X(t) = \mathbb{E}[e^{-tX}]$ of a random variable X.

Theorem 1

Let $\lambda, k > 0$ and X be a positive random variable with Laplace transform $\mathcal{L}_X$ satisfying $\mathbb{E}\big|X\,(d/dx f(x)|_X)/f(X)\big| < \infty$. Then X has a $W(\lambda, k)$-distribution if and only if

$$t\mathcal{L}_X(t) = \mathbb{E}\left[\frac{1}{X}\left(k\left(\frac{X}{\lambda}\right)^k - k + 1\right)\left(1 - \mathrm{e}^{-tX}\right)\right]$$

for each $t > 0$.

Furthermore we can estimate the LaPlace transform from the sample through

$$\widehat{\mathcal{L}}_X = \frac{1}{n}\sum_{i=1}^{n} e^{-tX_i}, \qquad t > 0.$$

Stein characterizations

With estimators $\widehat{\lambda}_n$ and $\widehat{k}_n$ of λ and k this leads to the test statistic

$$T_n = n \int_0^\infty \left| \frac{1}{n} \sum_{j=1}^n \frac{1}{X_j} \left(\widehat{k}_n \left(\frac{X_j}{\widehat{\lambda}_n} \right)^{\widehat{k}_n} - \widehat{k}_n + 1 \right) \left(1 - \mathrm{e}^{-tX_j}\right) - \frac{t}{n} \sum_{j=1}^n \mathrm{e}^{-tX_j} \right|^2 w(t) \mathrm{d}t$$

with an appropriate weight function $w : [0, \infty) \to [0, \infty)$ satisfying

$$\int_0^\infty (t^4 + 1) w(t) \mathrm{d}t < \infty.$$

A reasonable choice is $w_a^{(1)}(t) = \mathrm{e}^{-a|t|}$ or $w_a^{(2)}(t) = \mathrm{e}^{-at^2}$, $t \in [0, \infty)$, where $a > 0$ is some tuning parameter since the test statistic can be calculated explicitly in these cases.

$\rightarrow$ $\mathcal{H}_0$ is rejected for large values of T_n.

Limit distribution under $\mathcal{H}_0$

In our setting we consider a triangular array $X_{n,1}, \dots, X_{n,n}$, $n \in \mathbb{N}$, of row-wise i.i.d. random variables, defined on a common probability space $(\Omega, \mathscr{A}, \mathbb{P})$, with

$$X_{n,1} \sim W(\lambda_n, k_n), \quad k_n, \lambda_n > 0,$$

and

$$\lim_{n\to\infty} \lambda_n = \lambda_0 > 0, \qquad \lim_{n\to\infty} k_n = k_0 > 0.$$

Since the true parameters are unknown, we estimate them by $\widehat{\lambda}_n$ and $\widehat{k}_n$. We assume the linear representations

$$\sqrt{n}(\widehat{\lambda}_n - \lambda_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \psi_1(X_{n,j}, \lambda_n, k_n) + o_{\mathbb{P}}(1),$$

$$\sqrt{n}(\widehat{k}_n - k_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \psi_2(X_{n,j}\lambda_n, k_n) + o_{\mathbb{P}}(1),$$

with ψ_1 and ψ_2 satisfying some integrability assumptions.

Limit distribution under $\mathcal{H}_0$

The *Maximum likelihood estimators* are solutions to

$$\widehat{\lambda}_n = \left(\frac{1}{n}\sum_{i=1}^n X_{n,i}^{\widehat{k}_n}\right)^{1/\widehat{k}_n} \quad \text{and} \quad \frac{n}{\widehat{k}_n} + \sum_{i=1}^n \log X_{n,i} = \frac{n}{\sum_{i=1}^n X_{n,i}^{\widehat{k}_n}} \sum_{i=1}^n X_{n,i}^{\widehat{k}_n} \log X_{n,i}.$$

and satisfy

$$\sqrt{n}\begin{pmatrix}\widehat{\lambda}_n - \lambda_n \\ \widehat{k}_n - k_n\end{pmatrix} = \frac{1}{\sqrt{n}}\sum_{i=1}^n \mathcal{I}(\lambda_0, k_0)^{-1}\frac{d}{d(\lambda,k)}\log f(X_{n,i},\lambda,k)\bigg|_{(\lambda_n,k_n)} + o_{\mathbb{P}}(1),$$

where $\mathcal{I}(\lambda_0, k_0)$ is the Fisher information matrix.

The *Moment estimators* are solutions to

$$\widehat{k}_n = \frac{\pi}{\sqrt{6}}(S_n^2)^{-1/2} \quad \text{and} \quad \log\widehat{\lambda}_n = \overline{\log X} + \frac{\gamma}{\widehat{k}_n},$$

where γ is the Euler-Mascheroni constant and $S_n^2 = \frac{1}{n-1}\sum_{i=1}^n(\log X_{n,i} - \overline{\log X})^2$. It can be shown that

$$\sqrt{n}\begin{pmatrix}\widehat{\lambda}_n - \lambda_n \\ \widehat{k}_n - k_n\end{pmatrix} = \frac{1}{\sqrt{n}}\sum_{i=1}^n \begin{pmatrix}\lambda_n(\log X_{n,i} + \frac{3\gamma k_n}{\pi^2}(\log X_{n,i} - \log\lambda_n + \gamma/k_n)^2 + \frac{\gamma}{2k_n} - \log\lambda_n) \\ \frac{k_n}{2} - \frac{3k_n^3}{\pi^2}(\log X_{n,i} - \log\lambda_n + \gamma/k_n)^2\end{pmatrix} + o_{\mathbb{P}}(1),$$

Limit distribution under $\mathcal{H}_0$

Let $\mathscr{L}_w^2 = L^2([0,\infty), \mathcal{B}_{[0,\infty)}, w(t)\mathrm{d}t)$ be the Hilbert space of Borel-measurable functions $g : [0,\infty) \to \mathbb{R}$ satisfying

$$\|g\|^2 = \int_0^\infty g^2(t)w(t)\mathrm{d}t < \infty$$

with respect to a measurable positive weight function $w(\cdot)$. The scalar product on $\mathscr{L}_w^2$ is defined by

$$\langle g, h \rangle = \int_0^\infty g(t)h(t)w(t)\mathrm{d}t.$$

A $\mathscr{L}_w^2$-valued random element is a measurable function $V : \Omega \to \mathscr{L}_w^2$. Let $\mathbb{E}\|V\|^2 < \infty$. Its expectation is the unique $\mu \in \mathscr{L}_w^2$ such that

$$\langle \mu, g \rangle = \mathbb{E}\big[\langle V, g \rangle\big]$$

for every $g \in \mathscr{L}_w^2$.

Limit distribution under $\mathcal{H}_0$

The covariance operator $\Sigma : \mathscr{L}_w^2 \to \mathscr{L}_w^2$ is the unique positive self-adjoint nuclear operator such that

$$\langle \Sigma g, h \rangle = \mathbb{E}\big[\langle V, g\rangle \langle V, h\rangle\big], \qquad g, h \in \mathscr{L}_w^2.$$

A function $K(s,t), s,t > 0$ is a kernel of the covariance operator Σ if

$$\Sigma g = \int_0^\infty K(s,t) g(t) w(t) dt, \qquad g \in \mathscr{L}_w^2.$$

We call a $\mathscr{L}_w^2$-valued random element V *Gaussian* if $\langle V, g\rangle$ is one-dimensional Gaussian for every $g \in \mathscr{L}_w^2$. We can write

$$T_n = \|V_n\|^2,$$

where

$$V_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \left[\frac{1}{X_{n,j}} \left(\widehat{k}_n \left(\frac{X_{n,j}}{\widehat{\lambda}_n} \right)^{\widehat{k}_n} - \widehat{k}_n + 1 \right) \left(1 - \mathrm{e}^{-tX_{n,j}}\right) - t\mathrm{e}^{-tX_{n,j}} \right]$$

for $t \geq 0$. We can interpret $V_n : \Omega \to \mathscr{L}_w^2$ as a random element of $\mathscr{L}_w^2$.

Limit distribution under $\mathcal{H}_0$

Theorem 2

Under the triangular array introduced at the beginning of this section, we have

$$T_n = \| V_n \|^2 \xrightarrow{D} \| \mathcal{W} \|^2, \qquad \text{as} \quad n \to \infty.$$

Here, $\mathcal{W}$ is a centered Gaussian element of $\mathscr{L}_w^2$ with covariance operator Σ_{λ_0,k_0} whose kernel is given by $\mathbb{E}[W(t)W(s)]$, where

$$\begin{aligned} W(t) = & \frac{1}{X}\left(\left(\frac{X}{\lambda_0}\right)^{k_0} k_0 - k_0 + 1\right)(1-\mathrm{e}^{-tX}) - t\mathrm{e}^{-tX} \\ & - \psi_1(X,\lambda_0,k_0)\frac{k_0^2}{\lambda_0^{k_0+1}}\mathbb{E}\left[X^{k_0-1}(1-\mathrm{e}^{-tX})\right] \\ & + \psi_2(X,\lambda_0,k_0)\Bigg(\frac{k_0}{\lambda_0^{k_0}}\mathbb{E}\left[X^{k_0-1}\log(X/\lambda_0)(1-\mathrm{e}^{-tX})\right] \\ & \quad - \mathbb{E}\left[X^{-1}(1-\mathrm{e}^{-tX})\right] + \frac{1}{\lambda_0^{k_0}}\mathbb{E}\left[X^{k_0-1}(1-\mathrm{e}^{-tX})\right]\Bigg), \end{aligned}$$

and X has the Weibull distribution $W(\lambda_0, k_0)$.

Limit distribution under $\mathcal{H}_0$

Sketch of the proof.

Part 1
With some Taylor expansions and the linear representations of $\widehat{\lambda}_n$ and $\widehat{k}_n$ one can show that there exist row-wise i.i.d. $\mathscr{L}^2_w$-valued random elements $W_{n,j}, 1 \leq j \leq n$ such that

$$\left\| V_n(\cdot) - \frac{1}{\sqrt{n}} \sum_{j=1}^{n} W_{n,j}(\cdot) \right\|^2 = o_{\mathbb{P}}(1).$$

Part 2
A central limit theorem for Hilbert space valued triangular arrays yields

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} W_{n,j} \xrightarrow{D} \mathcal{W},$$

where $\mathcal{W}$ is a centered Gaussian random element with covariance operator Σ_{λ_0,k_0}. Slutzky's lemma then yields the claim.

Limit distribution under contiguous alternatives

Let $X_{n,1}, \ldots, X_{n,n}$, $n \in \mathbb{N}$, be a triangular array of row-wise i.i.d. random variables having Lebesgue density

$$g_n(x) = f(x)\big(1 + c(x)/\sqrt{n}\big), \qquad x \in [0, \infty).$$

Here, f is the $W(\lambda, k)$-density for some fixed $\lambda, k > 0$, $c : [0, \infty) \to \mathbb{R}$ is a measurable, bounded function satisfying $\int_0^\infty c(x) f(x) \mathrm{d}x = 0$.

Theorem 3

Under the stated assumptions, we have

$$T_n \xrightarrow{D} \|\mathcal{W} + \zeta\|^2 \qquad \text{as } n \to \infty,$$

where $\zeta \in \mathscr{L}_w^2$ satisfies $\langle \zeta, g \rangle = \mathbb{E}[\langle \eta(X, \cdot), g(\cdot) \rangle c(X)]$ for each $g \in \mathscr{L}_w^2$, where X has the Weibull distribution $W(\lambda, k)$, and

$$\begin{aligned}
\eta(x, s) =& \frac{1}{x}\left(\left(\frac{x}{\lambda}\right)^k k - k + 1\right)(1 - \mathrm{e}^{-sx}) - t\mathrm{e}^{-sx} - \psi_1(x, \lambda, k)\frac{k^2}{\lambda^{k+1}}\mathbb{E}\left[X^{k-1}(1 - \mathrm{e}^{-sX})\right] \\
&+ \psi_2(x, \lambda, k)\bigg(\frac{k}{\lambda^k}\mathbb{E}\left[X^{k-1}\log(X/\lambda)(1 - \mathrm{e}^{-sX})\right] \\
&- \mathbb{E}\left[X^{-1}(1 - \mathrm{e}^{-sX})\right] + \frac{1}{\lambda^k}\mathbb{E}\left[X^{k-1}(1 - \mathrm{e}^{-sX})\right]\bigg), \qquad x, s \geq 0
\end{aligned}$$

Consistency

We know that the asymptotic distribution of T_n is equal to $\|\mathcal{W}\|^2$, where $\mathcal{W}$ is a centered Gaussian random element with covariance operator Σ_{λ_0,k_0}. Moreover,

$$\|\mathcal{W}\|^2 \overset{D}{=} \sum_{i=1}^{\infty} \eta_i X_i,$$

where η_i are the eigenvalues of Σ_{λ_0,k_0} and X_i is a sequence of i.i.d. Gaussian random variables.

$\rightarrow$ Not used in practice.

Instead we apply a parametric bootstrap procedure:

Consider an i.i.d. sequence $(X_n)_{n\in\mathbb{N}}$ of copies of X, where X is a non-degenerate positive random variable satisfying $\mathbb{E}[X^m] < \infty$ and $\mathbb{E}[|\log X|X^m] < \infty$ for each $m \in \mathbb{N}$. Moreover, we assume that there are $\lambda_0, k_0 > 0$ such that

$$(\widehat{\lambda}_n, \widehat{k}_n) \xrightarrow{\text{a.s.}} (\lambda_0, k_0), \qquad \text{as} \quad n \to \infty.$$

Consistency

For $X_1, \ldots, X_n$ as above the parametric bootstrap includes the following steps:

First compute the estimators $\widehat{\lambda}_n = \widehat{\lambda}_n(X_1, \ldots, X_n)$ and $\widehat{k}_n = \widehat{k}_n(X_1, \ldots, X_n)$. Then:

- Generate another sample $X_1^*, \ldots, X_n^*$ of size n following the $W(\widehat{\lambda}_n, \widehat{k}_n)$-law.
- Estimate the parameters λ and k from $X_1^*, \ldots, X_n^*$ and calculate the test statistic T_n.

By repeating this procedure b times, we obtain $T_{n,1}^*, \ldots, T_{n,b}^*$. Given the nominal level $\alpha \in (0,1)$, we use the empirical $(1-\alpha)$-quantile of $T_{n,1}^*, \ldots, T_{n,b}^*$, i.e.,

$$c_{n,b}^*(\alpha) = \begin{cases} T_{b(1-\alpha):b}^*, & b(1-\alpha) \in \mathbb{N}, \\ T_{\lfloor b(1-\alpha)\rfloor+1:b}^*, & \text{otherwise} \end{cases}$$

as a critical value. The hypothesis H_0 is rejected if $T_n(X_1, \ldots, X_n) > c_{n,b}^*(\alpha)$. With Theorem 2 one can show that

$$\mathbb{P}(T_n > c_{n,b}^*) \overset{n,b\to\infty}{\longrightarrow} \alpha \quad \text{under } \mathcal{H}_0 \quad \text{and} \quad \mathbb{P}(T_n > c_{n,b}^*) \overset{n,b\to\infty}{\longrightarrow} 1 \quad \text{under } \mathcal{H}_1.$$

Simulation

For the simulation we compare the following tests

- Anderson-Darling (AD) (based on a comparison between empirical and theoretical cdf).
- Tiku-Singh (TS) (based on the normalized spacings).
- A test based on the sample skewness (ST).
- Ozturk-Korukog (OK) (based on the comparison of two different parameter estimators).
- $T_n^{(1)}$ with weight function $w_a^{(1)}(t) = \mathrm{e}^{-a|t|}$ and tuning parameter $a \in \{1, 2, 5\}$.
- $T_n^{(2)}$ with weight function $w_a^{(2)}(t) = \mathrm{e}^{-at^2}$ and tuning parameter $a \in \{1, 2, 5\}$.

The parameters λ and k are estimated with the maximum likelihood method.

Simulation

Alt.	AD	TS	ST	OK	$T_{n,1}^{(1)}$	$T_{n,2}^{(1)}$	$T_{n,5}^{(1)}$	$T_{n,1}^{(2)}$	$T_{n,2}^{(2)}$	$T_{n,5}^{(2)}$
$W(1, 0.9)$	5	6	6	5	5	4	5	5	5	5
$W(1, 1.5)$	6	5	5	5	5	6	5	5	6	5
$W(1, 3)$	6	5	5	5	5	5	5	5	5	5
$W(1/4, 1)$	5	5	5	5	6	6	6	6	5	5
$\Gamma(8, 1)$	25	**45**	38	38	16	24	31	25	31	35
$\Gamma(2, 1)$	10	15	11	11	11	9	7	6	4	7
$\Gamma(0.2, 1)$	48	19	56	**60**	31	24	19	0	0	0
$LN(0, 0.5)$	55	**80**	71	70	65	57	63	50	59	62
$LN(0, 0.8)$	55	**79**	72	68	62	55	65	44	57	66
$LN(0, 1.2)$	55	**79**	71	68	51	40	61	30	42	62
$i\Gamma(3, 1)$	92	**99**	97	95	92	94	93	94	93	92
$i\Gamma(1.5, 1)$	97	**100**	99	98	97	96	98	93	97	98
$GG1$	9	5	11	11	10	**13**	12	**13**	12	12
$GG2$	28	**49**	43	41	19	15	28	16	25	36
$AddW1$	**5**	4	**5**	4	**5**	**5**	**5**	**5**	**5**	**5**
$AddW2$	97	86	98	98	**99**	**99**	**99**	**99**	98	98
$P(0.5, 2)$	33	37	37	40	22	35	42	43	46	**48**
$P(1.5, 2.5)$	23	28	28	**30**	19	14	23	12	16	27
$IG(1, 1)$	80	**96**	90	85	66	82	84	85	86	84
$IG(1, 2)$	89	**99**	96	92	73	90	91	93	93	90

Table: Percentages of rejection ($n = 50$, 5000 replications, $b = 500$ bootstrap samples)

Real data example

We apply the new tests to data of failure stresses of single carbon fibers (in GPa). We investigate 4 different datasets (with respect to the fiber lengths 1mm, 10mm, 20mm and 50mm) with sample sizes between 57 and 70.

Failure stresses of single fibers are often associated with the so-called weakest-link hypothesis (the strength of a fiber can be represented by the minimum of independent strengths of sections).

	1mm	10mm	20mm	50mm
$T_{n,5}^{(1)}$	0.189	0.013	0.215	0.228
$T_{n,5}^{(2)}$	0.180	0.019	0.219	0.218

Table: p-values of failure stresses of single carbon fibers of the test statistics $T_{n,5}^{(1)}$ and $T_{n,5}^{(2)}$

Selected References

- Betsch, S., and B. Ebner. "Characterizations of continuous univariate probability distributions with applications to goodness-of-fit testing." arXiv preprint arXiv:1810.06226 (2018).
- Betsch, Steffen, and Bruno Ebner. "A new characterization of the Gamma distribution and associated goodness-of-fit tests." Metrika 82.7 (2019): 779-806.
- Ley, Christophe, and Yvik Swan. "Stein's density approach and information inequalities." Electronic Communications in Probability 18 (2013): 1-14.
- Krit, Meryam, Olivier Gaudoin, and Emmanuel Remy. "Goodness-of-fit tests for the Weibull and extreme value distributions: A review and comparative study." Communications in Statistics-Simulation and Computation 50.7 (2021): 1888-1911.

Thank you for your attention!