A categorical approach to partial group actions

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This work was done with the financial support given by FAPESC

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May 16th, 2023

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Assumptions

- Over the course of this presentation, we will assume G is a group with identity e, and $\mathscr{C}$ is a category with pullbacks.
- Whenever we are in the context of sets, we shall assume X and Y are sets. Otherwise, we shall assumed X and Y are objects in $\mathscr{C}$.

Partial action data on sets

Definition 1.1

A **partial action datum** of G on X is a pair $(\{X_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, such that

- For each $g \in G$, X_g is a subset of X;
- For each $g \in G$, α_g is a function $\alpha_g\colon X_{g^{-1}} \to X$.

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Partial actions on sets

Definition 1.2

A **partial action** [2] of G on X is a partial action datum $(\{X_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ of G on X that satisfies the following axioms:

1. $X_e = X$ and $\alpha_e = id_X$;
2. $\alpha_g(X_{g^{-1}} \cap X_h) \subseteq X_g \cap X_{gh}$, for all $g, h \in G$;
3. $\alpha_h \circ \alpha_g = \alpha_{hg}$ on $X_{g^{-1}} \cap X_{(hg)^{-1}}$, for all $g, h \in G$.

Examples

Let α be a global action of G on X, and consider the partial action datum $(\{X\}_{g\in G}, \{\alpha_g\}_{g\in G})$ of G on X. This partial action datum is a partial action of G on X

Let $G = \mathbb{Z}$ and $X = \mathbb{N}$. Consider the partial action datum of $\mathbb{Z}$ on $\mathbb{N}$

$$(\{X_z\}_{z\in\mathbb{Z}}, \{\alpha_z\}_{z\in\mathbb{Z}})$$

where when $z \geq 0$,

$$X_{-z} = \mathbb{N} \text{ and } \alpha_z : X_{-z} \to X \text{ maps } x \in \mathbb{N} \text{ to } x + z$$

and when $z < 0$,

$$X_{-z} = \{x \in \mathbb{N} : x \geq -z\} \text{ and } \alpha_z : X_{-z} \to X \text{ maps } x \in X_{-z} \text{ to } x + z.$$

This partial action datum is a partial action of $\mathbb{Z}$ on $\mathbb{N}$.

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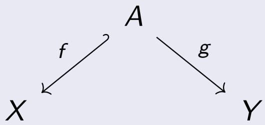
Partial morphisms

Definition 1.3

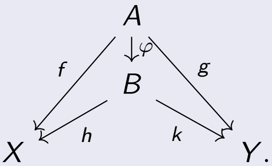
A **partial morphism** from X to Y is a triple (A, f, g), in which A is an object of $\mathscr{C}$, f is a monomorphism from A to X and g is a morphism from A to Y, as illustrated on the diagram



Morphisms between partial morphisms

Definition 1.4

Given (A, f, g) and (B, h, k) partial morphisms from X to Y, a morphism from (A, f, g) to (B, h, k) is a morphism $\varphi\colon A \to B$ on $\mathscr{C}$ such that the following diagram commutes.



With those morphisms, the class $\mathbf{Par}_{\mathscr{C}}(X, Y)$ of partial morphisms from X to Y form a category.

We shall denote by $[A, f, g]$ the isomorphism class represented by (A, f, g) in $\mathbf{Par}_{\mathscr{C}}(X, Y)$:

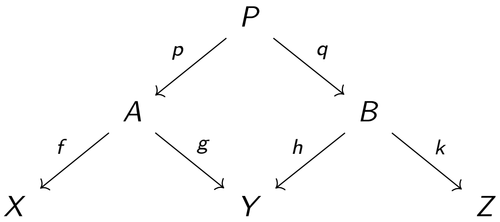
$$[A, f, g] = \{(B, h, k) \in \mathbf{Par}_{\mathscr{C}}(X, Y) : (B, h, k) \cong (A, f, g)\}.$$

And we shall denote by $\mathbf{par}_{\mathscr{C}}(X, Y)$ the set of isomorphism classes of partial morphisms from X to Y:

$$\mathbf{par}_{\mathscr{C}}(X, Y) = \{[A, f, g] : (A, f, g) \in \mathbf{Par}_{\mathscr{C}}(X, Y)\}$$

Composition of isomorphism classes of partial morphisms

Given $[A, f, g] \in \mathbf{par}_{\mathscr{C}}(X, Y)$ and $[B, h, k] \in \mathbf{par}_{\mathscr{C}}(Y, Z)$, the composition $[B, h, k] \bullet [A, f, g]$ of those isomorphism classes is given by the isomorphism class represented by the external partial morphism in the diagram



,

whose square is a pullback square. That is,

$$[B, h, k] \bullet [A, f, g] = [P, f \circ p, k \circ q].$$

Category of isomorphism classes of partial morphisms

We define the category $\mathbf{par}_{\mathscr{C}}$ to be the category whose objects are objects in $\mathscr{C}$ and, given two objects X and Y in $\mathscr{C}$, the set of morphisms from X to Y is $\mathbf{par}_{\mathscr{C}}(X, Y)$

Partial morphisms in Set

Proposition 1.5

Every isomorphism class $[A, f, g] \in \mathbf{par}_{\mathbf{Set}}(X, Y)$ *has exactly one representative* (B, ι, h) *where* $B \subseteq X$ *and* ι *is the inclusion of* B *on* X*.*

Thus, there is a bijection between $\mathbf{par}_{\mathbf{Set}}(X, Y)$ and the set of partial functions from X to Y.

Lemma 1.6 (Hu, Vercruysse, 2020 [4])

Let G be a group and X a set. There is a correspondence between

(1) *partial action data of G on X;*

(2) $\mathbf{par}_{\mathbf{Set}}(G \times X, X)$*;*

(3) *functions from G to* $\mathbf{par}_{\mathbf{Set}}(X, X)$*.*

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   - Motivations in the case of sets
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Partial action data on objects in categories

Definition 1.7

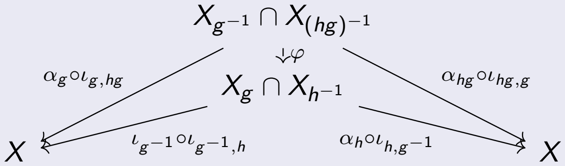
A **partial action datum** of G on $X \in \mathscr{C}$ is a function from G to $\mathbf{par}_{\mathscr{C}}(X, X)$.

Partial actions on objects in categories

Definition 1.8

A partial action datum $\alpha(g) = [X_{g^{-1}}, \iota_g, \alpha_g]$ of G on X is said to be a **partial action** of G on X if

1. $\alpha(e) = [X, id_X, id_X]$;
2. For all $g, h \in G$ there exists a morphism$X_{g^{-1}} \cap X_{(hg)^{-1}} \xrightarrow{\varphi} X_g \cap X_{h^{-1}}$ such that the following diagram commutes



Partial actions seen in the literature

- Every partial action of a group G on a ring X corresponds to a partial action of G on the object X in the category of rings, but because we ask the subsets of X to be ideals and not subrings, the converse fails to be true.
- Thus, partial actions in this categorical sense generalize properly partial actions on rings.
- Similarly, partial actions in this categorical sense generalize properly partial actions studied in the literature over other structures.

Global actions on objects in categories

Definition 1.9

A **global action** of G on $X \in \mathscr{C}$ is a group morphism $\alpha : G \to \operatorname{Aut} X$.

A global action α of G on X can be described as the partial action datum of G on X that maps $g \in G$ to $[X, id_X, \alpha(g)]$.
Any such partial action datum that comes from a global action is a partial action.

Inverse semigroups and partial actions

Outline

Definition 2.1

An **inverse semigroup** is a semigroup S such that for each $s \in S$ there exists a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$.

Partial bijections

Definition 2.2

Let X be a set. A **partial bijection** in X is a bijection between two subsets of X. The set of partial bijections in X is denoted by $\mathcal{I}(X)$.

We define a product between two partial bijections in X as their composition in the largest domain where it makes sense.
That is, if φ and ψ are partial bijections in X, then

$$\mathrm{dom}(\varphi\psi) = \psi^{-1}(\mathrm{ran}(\psi) \cap \mathrm{dom}(\varphi))$$

and

$$\mathrm{ran}(\varphi\psi) = \varphi(\mathrm{ran}(\psi) \cap \mathrm{dom}(\varphi)).$$

With this product, $\mathcal{I}(X)$ is an inverse semigroup.

A partial action on a set can be equivalently defined in terms of partial bijections as follows:

Definition 2.3

A **partial action** of G on X is a family $\Theta = \{\theta_g\}_{g \in G}$ formed by partial bijections in X such that

i) $\theta_g : X_{g^{-1}} \to X_g$ for each $g \in G$
ii) $\theta_e = id_X$;
iii) $\theta_h \theta_g \subseteq \theta_{hg}$, for all $g, h \in G$.

Theorem 2.4 (Exel, 1998 [3])

A function $\theta : G \to \mathcal{I}(X)$ corresponds to a partial action of G on X if, and only if,

1. $\theta(e) = id_X$,
2. $\theta(h)\theta(g)\theta(g^{-1}) = \theta(hg)\theta(g^{-1})$, *for all* $g, h \in G$.

Definition 2.5

Let G be a group. The **Exel's semigroup** constructed from G is the semigroup $\mathcal{S}(G)$ that is generated by the set $\{[g] : g \in G\}$, subject to the relations

1. $[g^{-1}][g][h] = [g^{-1}][gh]$
2. $[g][h][h^{-1}] = [gh][h^{-1}]$
3. $[g][e] = [g]$
4. $[e][g] = [g]$

for each $g, h \in G$, and in which e is the identity of G.

$\mathcal{S}(G)$ is an inverse semigroup in which $[g]^* = [g^{-1}]$.

Theorem 2.6 (Exel, 1998 [3])

There is a one-to-one correspondence between

1. *partial actions of G on X*
2. *identity preserving semigroup morphisms from $\mathcal{S}(G)$ to $\mathcal{I}(X)$.*

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Partial isomorphisms

Definition 2.7

A **partial isomorphism** from X to Y is a partial morphism (A, f, g) from X to Y such that g is a monomorphism.

We shall denote by $\mathbf{iso}_{\mathscr{C}}$ the subcategory of $\mathbf{par}_{\mathscr{C}}$ whose morphisms are isomorphism classes represented by partial isomorphisms.
We also shall denote by $\mathcal{I}(X)$ the set of endomorphisms of X in $\mathbf{iso}_{\mathscr{C}}$.

Proposition 2.8

Let $X \in \mathscr{C}$. Then $\mathcal{I}(X)$ is an inverse semigroup, in which

$$[A, f, g]^* = [A, g, f].$$

Proposition 2.9

Let α be a partial action of G on $X \in \mathscr{C}$. Then, $\alpha(g) \in \mathcal{I}(X)$ for all $g \in G$.

Partial actions and the natural partial order on $\mathcal{I}(X)$

Theorem 2.10

A partial action datum α of G on X is a partial action if, and only if,

- $\alpha(G) \subseteq \mathcal{I}(X)$,
- $\alpha(g^{-1}) = \alpha(g)^*$ *for every* $g \in G$,
- $\alpha(e)$ *is the identity of* $\mathcal{I}(X)$,
- $\alpha(h) \bullet \alpha(g) \leq \alpha(hg)$ *for every* $g, h \in G$.

Theorem 2.11

A partial action datum α of G on X is a partial action if, and only if,

- $\alpha(G) \subseteq \mathcal{I}(X)$,
- $\alpha(e)$ *is the identity of* $\mathcal{I}(X)$,
- $\alpha(h) \bullet \alpha(g) \bullet \alpha(g^{-1}) = \alpha(hg) \bullet \alpha(g^{-1})$ *for all* $g, h \in G$.

Theorem 2.12

Let G be a group and X an object of $\mathcal{C}$. There is a correspondence between the sets of

(1) *partial actions of G on X;*

(2) *identity preserving semigroup morphisms from $\mathcal{S}(G)$ to $\mathcal{I}(X)$.*

Induced partial actions and globalizations

Outline

Induced partial actions

Given a global action α of G on X, and S a subset of X, consider the partial action datum

$$(\{S_g\}_{g\in G}, \{\beta_g\}_{g\in G}),$$

where, for each $g \in G$,

$$S_{g^{-1}} = S \cap \alpha_g^{-1}(S)$$

and $\beta_g : S_{g^{-1}} \to S$ is given by $\beta_g(s) = \alpha_g(s)$ for each $s \in S_{g^{-1}}$.
This partial action datum is a partial action of G on S, which is called the **induced partial action** of α on S.

Globalizations

Definition 3.1

A **globalization** for a partial action β of G on S is a global action α of G on a set X containing S such that β is the induced partial action of α on S.

Partial action datum morphisms

Definition 3.2

Given two partial action data $\alpha = (\{X_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ of G on X and $\beta = (\{Y_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ of G on Y, a **partial action datum morphism**, or, in short, **datum morphism**, from α to β is a function $f\colon X \to Y$ such that

1. $f(X_g) \subseteq Y_g$ for all $g \in G$
2. $f \circ \alpha_g = \beta_g \circ f$ on $X_{g^{-1}}$ for all $g \in G$.

The category of partial action data

Definition 3.3

The category G-**Datum**, or the **category of partial action data of** G on sets, is the category whose objects are partial action data of G on sets, and whose morphisms are datum morphisms, in which the composition of two morphisms is given by the usual composition of functions.

The categories of partial actions and global actions

Definition 3.4

The category G-**pAct**, or the **category of partial actions of** G on sets, is the full subcategory of G-**Datum** whose objects are the partial actions of G on sets.

Definition 3.5

The category G-**Act**, or the **category of (global) actions of** G on sets, is the full subcategory of G-**Datum** whose objects are the global actions of G on sets.

Theorem 3.6 (Abadie, 2003 [1])

*Let β be a partial action of G on a set S. Then, β has a globalization α of G on a set X such that the inclusion of S on X is a datum morphism between β and α that is a reflection of β in G-**Act**.*

Such a globalization is called the **enveloping action** of β.

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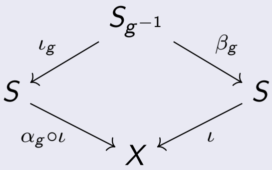
Induced partial actions

Definition 3.7

Let α be a global action of G on an object $X \in \mathscr{C}$ and $\iota : S \to X$ be a monomorphism in $\mathscr{C}$. Let $\beta(g) = [S_{g^{-1}}, \iota_g, \beta_g]$ be the partial action datum of G on S where for each $g \in G$ the following diagram is a pullback.



β is a partial action of G on S, called the **induced partial action** of α on S (via the monomorphism ι).

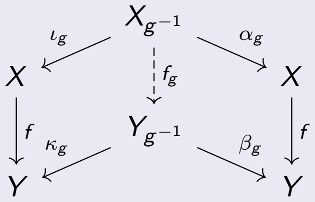
Definition 3.8

A **globalization** for a partial action β is a pair (α, ι) in which α is a global action of G on an object X in $\mathscr{C}$ and ι is a monomorphism from S to X such that β is the induced partial action of α on S via ι.

Partial action datum morphisms

Definition 3.9

Let α and β be partial action data of G on X and Y, with, say, $\alpha(g) = [X_{g^{-1}}, \iota_g, \alpha_g]$ and $\beta(g) = [Y_{g^{-1}}, \kappa_g, \beta_g]$ for all $g \in G$. A **partial action datum morphism**, or **datum morphism**, between α and β is a morphism $f : X \to Y$ such that for all $g \in G$ there is a morphism $f_g : X_{g^{-1}} \to Y_{g^{-1}}$ such that the following diagram commutes:



The category of partial action data

Definition 3.10

The category G-$\mathbf{Datum}_{\mathscr{C}}$, or the **category of partial action data of** G on objects in $\mathscr{C}$, is the category whose objects are partial action data of G on objects in $\mathscr{C}$, and whose morphisms are datum morphisms, in which the composition of two morphisms is given by the usual composition of the morphisms in $\mathscr{C}$.

Definition 3.11

The category G-$\mathbf{pAct}_{\mathscr{C}}$, or the **category of partial actions of** G on objects in $\mathscr{C}$, is the full subcategory of G-**Datum** whose objects are the partial actions of G on objects in $\mathscr{C}$.

Definition 3.12

The category G-$\mathbf{Act}_{\mathscr{C}}$, or the **category of (global) actions of** G on objects in $\mathscr{C}$, is the full subcategory of G-**Datum** whose objects are the global actions of G on objects in $\mathscr{C}$.

Universal globalizations

Definition 3.13

A **universal globalization** for a partial action β is a pair (α, ι) such that

1. (α, ι) is a globalization for β.
2. ι is a datum morphism between β and α that is a reflection of β in G-$\mathbf{Act}_{\mathscr{C}}$.

Outline

Assumptions

- For the remainder of this presentation, β will be a partial action of G on an object S in $\mathscr{C}$, in which $\beta(g) = [S_{g^{-1}}, \iota_g, \beta_g]$ for all $g \in G$.

Universal globalizations and globalizations

Theorem 3.14

Assume β has a reflection $\iota : \beta \to \alpha$ in G-$\mathbf{Act}_{\mathscr{C}}$. Then, the following are equivalent:

- *β has a universal globalization,*
- *β has a globalization,*
- *For each $g \in G$ the diagram*

$$\begin{array}{ccccc} & & S_{g^{-1}} & & \\ & \swarrow^{\iota_g} & & \searrow^{\beta_g} & \\ S & & & & S \\ & \searrow_{\alpha_g \circ \iota} & & \swarrow_{\iota} & \\ & & X & & \end{array}$$

is a pullback diagram in $\mathscr{C}$.

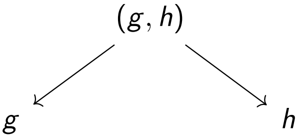
Universal globalizations and globalizations

Theorem (Continuation of Theorem 3.6)

In this case, (α, ι) is a universal globalization for β.

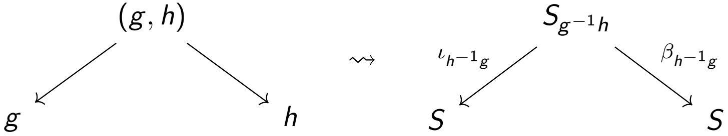
Existence of a reflection - colimit

Consider the category I whose class of objects is the set $(G \times G) \cup G$, in which for all $g, h \in G$ there is a single morphism between (g, h) and g, and between (g, h) and h, and there are no other non trivial morphisms, as illustrated:



Existence of a reflection - colimit

Given the partial action β, let F be the functor from I to $\mathscr{C}$ that sends $(g, h) \in (G \times G) \subseteq I$ to $S_{g^{-1}h}$ and $g \in G \subseteq I$ to S, and, given $g, h \in G$, that sends the unique morphism between (g, h) and g to $S_{g^{-1}h} \overset{\iota_{h^{-1}g}}{\longrightarrow} S$, and the unique morphism between (g, h) and h to $S_{g^{-1}h} \overset{\beta_{h^{-1}g}}{\longrightarrow} S$, as illustrated:



Existence of a reflection - colimit

Theorem 3.15

Assume the colimit $\eta : \mathscr{F} \to \Delta(K)$ (in which $\Delta(K)$ is the functor that is constant and equal to $K \in \mathscr{C}$) of the functor F exists. Then, β has a reflection in G-$\mathbf{Act}_{\mathscr{C}}$.
In this case, there exists a global action α of G on K constructed from this colimit such that η_e is a datum morphism between β and α that is a reflection of β in G-$\mathbf{Act}_{\mathscr{C}}$.

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Existence of a reflection - coproducts and coequalizer

Given the partial action β, assume the coproducts $\coprod_{g\in G} S$ and $\coprod_{(g,h)\in G\times G} S_{g^{-1}h}$ exist in $\mathscr{C}$, in which the associated inclusion morphisms are, respectively, u_g for $g \in G$ and $u_{(g,h)}$ for $(g,h) \in G \times G$.
Consider the morphisms

$$p = \coprod_{(g,h)\in G\times G} S_{g^{-1}h} \xrightarrow{\coprod(u_g \circ \iota_{h^{-1}g})} \coprod_{g\in G} S$$
$$q = \coprod_{(g,h)\in G\times G} S_{g^{-1}h} \xrightarrow{\coprod(u_h \circ \beta_{h^{-1}g})} \coprod_{g\in G} S$$

In this case, a coequalizer for p and q induces a colimit for the functor F associated with β.

Existence of a reflection - coproducts and coequalizer

Corollary 3.16

Assume the coproducts $\coprod_{g\in G} S$ and $\coprod_{(g,h)\in G\times G} S_{g^{-1}h}$ exist $\mathscr{C}$, and suppose that p and q have a coequalizer $\coprod_{g\in G} S \xrightarrow{c} K$. Then, β has a reflection in G-$\mathbf{Act}_{\mathscr{C}}$.
In this case, there exists a global action α of G on K constructed from this coequalizer such that $c \circ u_e$ is a datum morphism between β and α that is a reflection of β in G-$\mathbf{Act}_{\mathscr{C}}$.

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Existence of a reflection - coproducts and coequalizer in G-$\mathbf{Act}_{\mathscr{C}}$

- Given the partial action β, we keep the assumption that the coproducts $\coprod_{g\in G} S$ and $\coprod_{(g,h)\in G\times G} S_{g^{-1}h}$ exist in $\mathscr{C}$.
- There are certain global actions ψ and φ of G acting, respectively, on $\coprod_{g\in G} S$ and $\coprod_{(g,h)\in G\times G} S_{g^{-1}h}$, which are induced naturally due to the structure of G and the objects being coproducts.
- The morphisms p and q are, in fact, datum morphisms from φ to ψ.

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Existence of a reflection - coproducts and coequalizer in G-$\mathbf{Act}_{\mathscr{C}}$

Theorem 3.17

Assume the coproducts $\coprod_{g\in G} S$ e $\coprod_{(g,h)\in G\times G} S_{g^{-1}h}$ exist $\mathscr{C}$. Then, β has a reflection in G-$\mathbf{Act}_{\mathscr{C}}$ if, and only if, p and q have a coequalizer $\psi \xrightarrow{c} \alpha$ in G-$\mathbf{Act}_{\mathscr{C}}$.
In this case, $c \circ u_e$ is a datum morphism between β and α that is a reflection of β in G-$\mathbf{Act}_{\mathscr{C}}$.

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