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A gentle introduction of the Connes-Moscovici's bialgebroid and its universal properties

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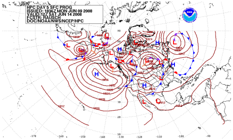
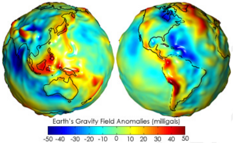
Geometry	**Algebra**
Smooth manifolds M	Smooth functions on the manifold $\mathcal{C}(M) = \{f : M \to \mathbb{R} \mid f \text{ smooth}\}$





Geometry	Algebra
Tangent space to M at a point p: T_pM	p-derivations on $\mathcal{C}(M)$
	$X_p : \mathcal{C}(M) \to \mathbb{R}$ such that $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$ for all $f, g \in \mathcal{C}(M)$

From differential geometry to commutative algebra

Geometry		**Algebra**
$TM = \bigcup_{p \in M} T_p M$	Tangent bundle	Derivations of $\mathcal{C}(M)$ $\mathrm{Der}(\mathcal{C}(M))$
$\begin{tikzcd} TM \arrow[r, "\pi"'] & M \arrow[l, bend right, dotted, "X"'] \end{tikzcd}$	Vector fields $\mathfrak{X}(M)$	$X : \mathcal{C}(M) \to \mathcal{C}(M)$ such that $X(f \cdot g) = X(f) \cdot g + f \cdot X(g)$ for all $f, g \in \mathcal{C}(M)$
		$[X, Y] := X \circ Y - Y \circ X$ is in $\mathrm{Der}(\mathcal{C}(M))$

$$[X, X] = 0 \quad \text{and} \quad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Lie algebras

Definition

A Lie algebra is a vector space L together with a bilinear map $[\text{-},\text{-}] : L \times L \to L$ such that $[X, X] = 0$ and

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] = 0 \qquad (\textit{Jacobi identity})$$

for all $X, Y, Z \in L$.

Facts

- $\mathfrak{X}(M)$ is a Lie algebra with $[X, Y]_p(f) = X_p(Y(F)) - Y_p(X(f))$.
- For any algebra A, $\mathrm{Der}(A)$ is a Lie algebra with $[X, Y] = X \circ Y - Y \circ X$.
- Any algebra A with $[a, b] = ab - ba$ is a Lie algebra.

Universal enveloping algebras

Definition

A universal enveloping algebra for a Lie algebra L is an associative algebra $U(L)$ together with a morphism of Lie algebras $L \xrightarrow{\iota} U(L)$ which is universal with respect to this property.

Remark

$(U(L), \iota)$ is a universal arrow from L to the functor $\mathcal{L} : \mathsf{Alg}_{\Bbbk} \to \mathsf{Lie}_{\Bbbk}$.

Facts

- $$\Delta : U(L) \to U(L) \otimes U(L), \qquad X \mapsto X \otimes 1 + 1 \otimes X,$$
  $$\text{and} \qquad \varepsilon : U(L) \to \Bbbk, \quad X \mapsto 0,$$
  make of $U(L)$ a bialgebra.
- For B a bialgebra, the set $P(B) := \{b \in B \mid \Delta(b) = b \otimes 1 + 1 \otimes b\}$ is a Lie algebra. $(U(L), \iota)$ is also a universal arrow from L to the functor $P : \mathsf{Bialg}_{\Bbbk} \to \mathsf{Lie}_{\Bbbk}$.

From differential geometry to commutative algebra

Geometry	Algebra
Vector bundles and global sections	Finitely generated and projective $\mathcal{C}(M)$-modules

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ M \arrow[u, bend left, dotted, "X"] \end{tikzcd}$$

E smooth

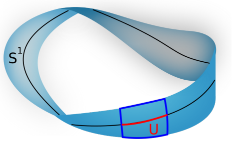
π smooth

$\pi^{-1}(\{p\}) \cong V$

$\pi^{-1}(U) \cong U \times V$

(V fixed f.d. vector space)

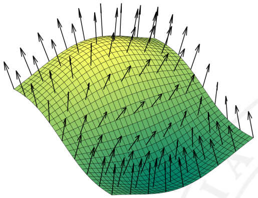
$\Gamma(E) =$ global sections



$f \in \mathcal{C}(M)$, $X \in \Gamma(E)$

$(f \cdot X)(p) := f(p)X_p$

for all $p \in M$



From differential geometry to (commutative) algebra

Geometry	**Algebra**
Lie algebroids	Lie-Rinehart algebras
vector bundle $A \xrightarrow{\varpi} M$	$\mathcal{C}(M)$-module L
vector bundle morphism $A \xrightarrow{a} TM$	morphism of $\mathcal{C}(M)$-modules $L \xrightarrow{\omega} \mathrm{Der}\big(\mathcal{C}(M)\big)$
$[\text{-},\text{-}] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$	$[\text{-},\text{-}] : L \times L \to L$
$\Gamma(A)$ Lie algebra $a([X,Y]) = \big[a(X), a(Y)\big]$ $\big[X, fY\big] = f\big[X,Y\big] + a(X)(f)Y$	L Lie algebra ω of Lie algebras $\big[X, fY\big] = f\big[X,Y\big] + X(f)Y$

Universal enveloping A-rings

Definition

A universal enveloping A-ring for a Lie-Rinehart algebra L is an associative A-ring U with unit $A \xrightarrow{u} U$ together with a morphism of Lie algebras $L \xrightarrow{\iota} U$ satisfying

$$u(a)\iota(X) = \iota(aX) \quad \text{and} \quad \iota(X)u(a) - u(a)\iota(X) = u(X(a)) \quad \forall\, a \in A, X \in L$$

and which is universal with respect to these properties.

Facts

- **Rinehart**: a UE A-ring for L exists - $\mathcal{U}(L)$.
- $$\Delta : \mathcal{U}(L) \to \mathcal{U}(L) \otimes_A \mathcal{U}(L), \qquad X \mapsto X \otimes_A 1 + 1 \otimes_A X,$$
  $$\varepsilon : \mathcal{U}(L) \to A, \qquad X \mapsto 0,$$
  make of $\mathcal{U}(L)$ a cocommutative A-bialgebroid.
- For $\mathcal{B}$ an A-bialgebroid, $\mathcal{P}(\mathcal{B}) := \{b \in \mathcal{B} \mid \Delta(b) = b \otimes_A 1 + 1 \otimes_A b\}$. If $\mathcal{B}$ is cocommutative, then $\mathcal{P}(\mathcal{B})$ is a Lie-Rinehart algebra. $(\mathcal{U}(L), \iota)$ is a universal arrow from L to the functor $\mathcal{P} : \mathrm{CCBialgd}_A \to \mathrm{LieRin}_A$.

From Lie-Rinehart to anchored Lie algebras

Anchored Lie algebras

Fix: $\Bbbk$ field of $\mathrm{char}(\Bbbk) = 0$ A a non-comm $\Bbbk$-algebra

Definition

An A-anchored Lie algebra is a Lie algebra L over $\Bbbk$ together with a morphism of Lie algebras $L \xrightarrow{\omega} \mathrm{Der}(A)$.

Definition

A universal enveloping A^{e}-ring for an A-anchored Lie algebra is an associative A^{e}-ring U with unit $\eta : A^{\mathrm{e}} \to U$ together with a morphism of Lie algebras $L \xrightarrow{\jmath} U$ satisfying

$$\Big[\jmath(X), \eta(a \otimes b)\Big] = \eta\big(X \cdot (a \otimes b)\big) \qquad \text{for all } a, b \in A, X \in L$$

and which is universal with respect to these properties.

Facts

- A is a representation of L $\quad\rightsquigarrow\quad$ A is a $U(L)$-module algebra.
- $$\begin{aligned} \Big(A \otimes U(L) \otimes A\Big) \otimes \Big(A \otimes U(L) \otimes A\Big) &\to \Big(A \otimes U(L) \otimes A\Big) \\ (a \otimes u \otimes b) \otimes (a' \otimes u' \otimes b') &\mapsto \sum a(u_1 \cdot a') \otimes u_2 u' \otimes (u_3 \cdot b')b \end{aligned}$$

  $$\begin{aligned} \Delta : \Big(A \otimes U(L) \otimes A\Big) &\to \Big(A \otimes U(L) \otimes A\Big) \otimes_A \Big(A \otimes U(L) \otimes A\Big) \\ (a \otimes u \otimes b) &\mapsto \sum (a \otimes u_1 \otimes 1) \otimes_A (1 \otimes u_2 \otimes b) \end{aligned}$$

  $$\varepsilon : \Big(A \otimes U(L) \otimes A\Big) \to A, \qquad (a \otimes u \otimes b) \mapsto a\varepsilon(u)b$$

  makes of $A \otimes U(L) \otimes A$ an A-bialgebroid $A \odot U(L) \odot A$.

Theorem (S. '20)

The Connes-Moscovici's bialgebroid $A \odot U(L) \odot A$ is the universal enveloping A^e-ring of the A-anchored Lie algebra L.

Remark

For $\mathcal{B}$ an A-bialgebroid, $\mathcal{P}(\mathcal{B}) := \{b \in \mathcal{B} \mid \Delta(b) = b \otimes_A 1 + 1 \otimes_A b\}$ is an A-anchored Lie algebra.

Theorem (S. '20)

$(A \odot U(L) \odot A, \jmath)$ is a universal arrow from L to the functor $\mathcal{P} : \mathsf{Bialgd}_A \to \mathsf{AnchLie}_A$.

Many thanks