Fractals, radix representation and automata

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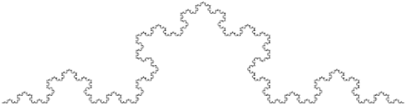
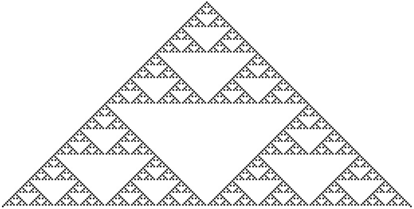
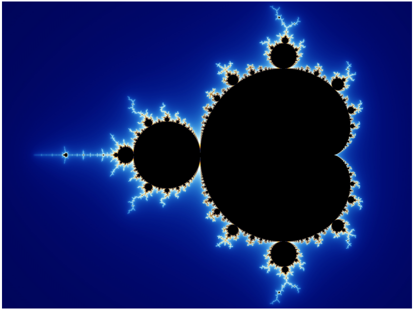
November 25, 2020

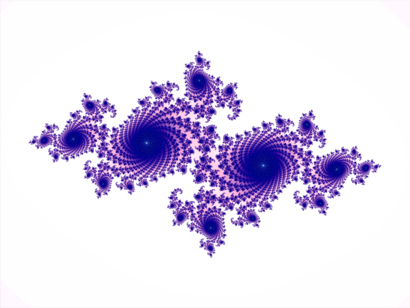
Outline

Mathematical Monsters



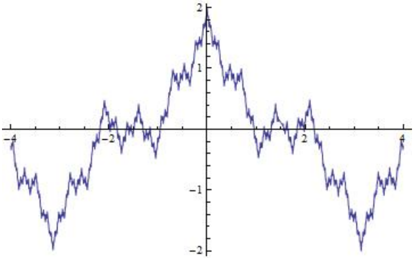




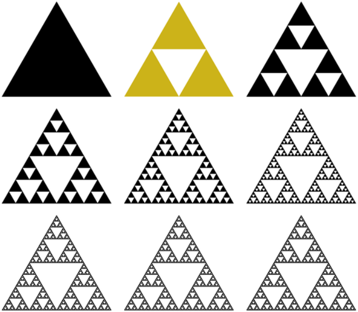


What is a fractal?

- Originally found as counter examples (Weierstrass function)
- Isolated works by Cantor, Julia...
- Benoît B. Mandelbrot found fractal geometry late 1970s
- Fractals have a great complexity, details in comp. to smooth, linear objects
- Many fractals have selfsimilar features
- However: in modern research this is not the defining quality!



Sierpinsky gasket I

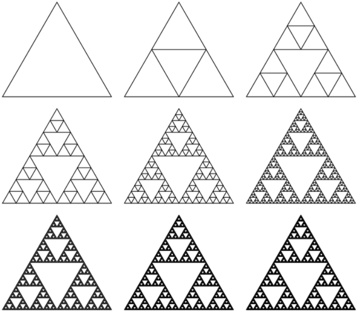


Approximating the **Sierpinsky triangle** by cutting out full triangles. What is its area $A(S_\infty)$?

$$A(S_0) := A_0, A(S_1) = \frac{3}{4}A_0, A(S_2) = \frac{3}{4}A(S_1) = \left(\frac{3}{4}\right)^2 A_0, \ldots$$

$$A(S_n) = \left(\frac{3}{4}\right)^n A_0, \ldots, A(S_\infty) = \lim_{n\to\infty}\left(\frac{3}{4}\right)^n A_0 = 0$$

Sierpinsky gasket II



Approx. the Sierpinsky adding triangle outlines. What is its length $L(S_\infty)$?

$$L(S_0) := L_0, L(S_1) = \frac{4}{3} L_0 \ldots$$

$$L(S_n) = \left(\frac{4}{3}\right)^n L_0, \ldots, L(S_\infty) = \lim_{n\to\infty} \left(\frac{4}{3}\right)^n L_0 = \infty$$

The Sierpinsky gasket has area 0, and length ∞: What is its dimension?

Dimension theory

Dimension theory deals with the development of purely topological notions of dimension. Demands on a proper dimension function are: [6]

(1) The d-dimensional unit hyper cube in $\mathbb{R}^d$ has $\dim = d$.

(2) (*Monotonicity*) If $X \subseteq Y$, $\dim(X) \leq \dim(Y)$.

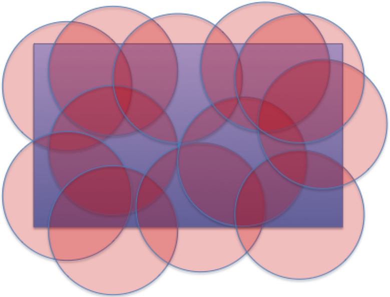
(3) (*Countable Stability*)

$$\dim\left(\bigcup_{j=1}^{\infty} X_j\right) = \sup_{j \geq 1} \dim(X_j). \tag{1}$$

(4) (*Invariance*) For $\mathcal{F}$ a subfamily of the homeomorphisms of $\mathbb{R}^n$ to $\mathbb{R}^n$, dim shall be invariant, i.e. for $\psi \in \mathcal{F}$

$$\dim(\psi(X)) = \dim(X). \tag{2}$$

Covering dimension



Let (X, τ) be a topological space and $\mathcal{U}$ an arbitrary open covering. We now search for refinements of $\mathcal{U}$ such that maximal $(n+1) \in \mathbb{N}$ sets intersect eachother simultaneously. The minimal number $\mathrm{Cov}(X)$, such that such a refinement exists for all coverings is called **Covering dimension**.

Theorem

Cov *satisfies all conditions and is topologically invariant*

Hausdorff dimension I

General idea: A set is s-dimensional, if it has non trivial s-dimensional Volume.

Definition and Proposition

Let (X, d) be a compact metric space and $\mathcal{U} = \{U_i, i \in I\}$ be an open cover of $A \subseteq X$. $\mathcal{U}$ is called a δ-**cover**, iff $0 \leq \mathrm{diam}(U_i) \leq \delta$. Define

$$\mathcal{H}^s_\delta(A) := \inf\left\{\sum_{i=1}^{\infty} \mathrm{diam}(U_i)^s \middle| U_i \text{ is a } \delta\text{-cover of } A\right\} \tag{3}$$

and the s-**dimensional Hausdorff measure** of A as

$$\mathcal{H}^s(A) := \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(A). \tag{4}$$

Hausdorff dimension II

Now the important observation is that viewed as a function in s, with values in $[0, \infty]$ the Hausdorff measure only has one point of discontinuity:

Lemma

Let $A \subseteq X$ and $s < t, \delta > 0$, then $\mathcal{H}^t_\delta(A) \leq \delta^{t-s}\mathcal{H}^s_\delta(A)$. Further if $\mathcal{H}^s(A) \leq \infty$, then $\mathcal{H}^t(A) = 0$ and if $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = \infty$.

Definition

Let (X, d) be a metric space and $A \subseteq X$. The **Hausdorff dimension** of A is defined as

$$\dim(A) := \sup\{s : \mathcal{H}^s(A) = \infty\} = \inf\{s : \mathcal{H}^s(A) = 0\}. \quad (5)$$

Fractals

Theorem

dim *satisfies all conditions and is invariant under bi-Lipschitz functions*

Theorem ([3])

Let $A \subseteq X$ *with* (X, d) *a metric space. Then* $\mathrm{Cov}(A) \leq \dim(A)$*.*

We are now able to define a fractal.

Definition

Let $A \subseteq X$ with (X, d) a metric space. Then A is **fractal**, iff $\mathrm{Cov}(A) < \dim(A)$.

Examples:

- Sierpinsky gasket: $\dim(S_\infty) = \frac{\log 3}{\log 2} \approx 1.585$
- Koch curve: $\dim(K_\infty) = \frac{\log 4}{\log 3} \approx 1.262$

Marstrand theorem

How does the dimension of a set in $\mathbb{R}^2$ changes by intersetion with a line?

Theorem ([4])

Let E be a Borel set in $\mathbb{R}^2$ and L_x the line, which is parallel to the y-axix through $(x, 0)$. It holds, that

$$\dim(E \cap (L_x)) \leq \max\{\dim(E) - 1, 0\} \tag{6}$$

for Lebesgue-almost all $x \in \mathbb{R}$.

On the other hand we get have.

Theorem ([4])

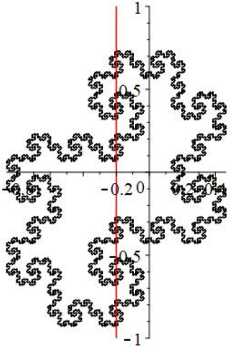
Let E be a Borel set in $\mathbb{R}^2$ and L_x the line, which is parallel to the y-axix through $(x, 0)$. It holds, that

$$\dim(E \cap (L_x)) \geq \dim(E) - 1, \tag{7}$$

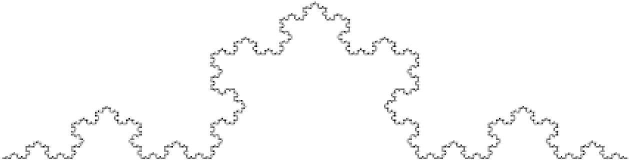
for all $x \in I$, where $I \subset \mathbb{R}$ has positive Lebesgue-measure.

Search for Marstrand dimension

- Combining the two theorems + rotation invariance of dim: Hausdorff dimension reduces by 1 for a large family of cases (uncountable, positive Lebesgue measure).
- In practice the exception cases are much easier to find.
- Except specifically constructed examples full filling *Marstrand dimension* not one example has been found.



Infinite details



Self-similar sets

Definition

Let (X, d) be a metric space and $(f_i)_{i=1}^m : X \to X$ be contractions.
$\emptyset \neq S \subseteq X$ is called **self-similar** iff

$$S = \bigcup_{i=1}^{m} f_i(S). \tag{8}$$

Iff the f_i are similar contractions, S is called a **self-affine** set. We call $\{f_i\}$ an **iterated function system (IFS)** and S its **attractor**.

Note: not all fractals are self-similar and not all self-similar sets are fractals!

Existence of self-similar sets

Given a family of contractions $\{f_i\}_{i=1}^m$ does such an attractor allways exists? The answer is yes! Use **Banach Fixed Point Theorem**

Theorem

Let (X, d) be a non-empty complete metric space with a contraction $T : X \to X$. Then T admits a unique fixed-point x in X: $T(x) = x$.

In the context of selfsimilar sets we show:

Theorem

Let $\mathcal{K}(\mathbb{R}^n)$ be the set of all non-empty compact subsets of X and d_H the Hausdorff metric. Then $(\mathcal{K}(\mathbb{R}^n), d_H)$ is a complete metric space. Furthermore given $(f_i)_{i=1}^m : \mathbb{R}^n \to \mathbb{R}^n$ contractions, then the map $F = \bigcup_i f_i : \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$, called the **Hutchinson operator**, *is a contraction.*

Construction self-similar sets

Recall the proof of Banach Fixed point Theorem:

Proposition

x can be found as follows: start with an arbitrary element $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_n := T(x_{n-1})$ for $n \geq 1$. Then $\lim_{n\to\infty} x_n = x$.

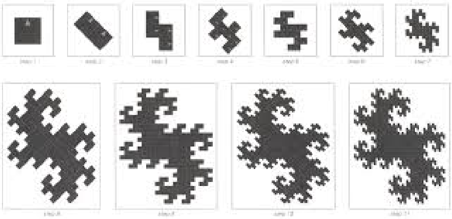
So no matter with what non empty compact set you start by applying the Hutchinson operator successively you can approximate the self-similar attractor.

The Knuth Twin Dragon

Now consider $\{f_0, f_1\} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f_0(x) = B^{-1}x$, $f_1(x) = B^{-1}(x + (1,0))$ and

$$B = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}. \tag{9}$$

The matrix B^{-1} describes a rotation of 45° and a contraction of $|\det(B)|^{-1} = 0.5$.



The Knuth Twin Dragon

Proposition ([2])

Let $\mathcal{K}$ *be the Twin Dragon, that is the attractor of the IFS* $\{f_0, f_1\}$ *with* $f_0(x) = B^{-1}x, f_1(x) = B^{-1}(x + (1,0))$ *and*

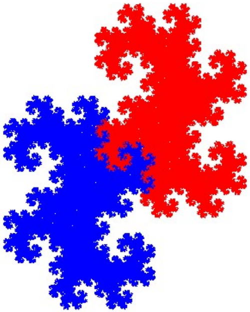
$$B = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}. \tag{10}$$

Then $\dim(\mathcal{K}) = 2$,

$$\dim(\partial\mathcal{K}) = \frac{\log\left(\frac{\sqrt[3]{3\sqrt{87}+28}}{3} + \frac{1}{3\sqrt[3]{3\sqrt{87}+28}} + \frac{1}{3}\right)}{\log\sqrt{2}} = 1.523627\ldots \tag{11}$$

Address of a point

On the other hand the Knuth Twin dragon $\mathcal{K}$ is the fixed point of $F = f_0 \cup f_1$. So $\mathcal{K}$ dissolves into the images of f_0 and f_1.



Applying F once again we have a four pieces of $\mathcal{K}$:

$$f_0(f_0(\mathcal{K})), f_0(f_1(\mathcal{K})), f_1(f_0(\mathcal{K})), f_1(f_1(\mathcal{K})),$$

By separating $\mathcal{K}$ further and further we can assign to every point $x \in \mathcal{K}$ and **address**, that is an infinite sequence $\omega = (d_1, d_2, d_3, \ldots) \in \{0,1\}^*$

Decimal-, Binary-, (-10)-ary numbers?

It is a well know fact that for any base $b \in \mathbb{N} \setminus \{0\}$ every natural number γ can be represented uniquely in the form

$$\gamma = \sum_{j=0}^{n} d_j b^j \tag{12}$$

with digits $d_j \in \{0, \dots, b-1\}$. What happens if we also allow negative bases?

$$\begin{array}{cccc} & 2 & -10 & -2 \\ 10 & 2 & -10 & -2 \\ 100 & 4 & 100 & 4 \\ 110 & 6 & 90 & 2 \\ 11 & 3 & -9 & -1 \end{array} \tag{13}$$

Choosing negative bases $b \in \mathbb{Z}$ and digits $\{0, \dots, |b|\}$ allows us to uniquely represent all **integers**!

Complex bases

What happens if we allow complex bases?

$$\begin{array}{ccccc} & 2 & -10 & -2 & -i-1 \\ 10 & 2 & -10 & -2 & -i-1 \\ 100 & 4 & 100 & 4 & -2 \\ 110 & 6 & 90 & 2 & -i-3 \\ 11 & 3 & -9 & -1 & -i \end{array} \qquad (14)$$

When is the set of representable numbers $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$?

Canonical Number Systems (CNS)

Definition (Radix Representation)

Let $P(x) = x^m + b_{m-1}x^{m-1} + \ldots + b_0 \in \mathbb{Z}[x]$ monic polynomial with $m \geq 1$. Let $\mathcal{R} = \mathbb{Z}[x]/P(x)\mathbb{Z}[x]$ and $\mathcal{N}$ a complete residue system $\mod |b_0|$. The pair $(P(x), \mathcal{N})$ is called a **NS** in $\mathcal{R}$, if each $\gamma \in \mathcal{R}$ admits a unique representation of the form

$$\gamma = \sum_{j=0}^{n} d_j x^j \tag{15}$$

where $d_j \in \mathcal{N}, n \in \mathbb{N}$ minimal, such that $d_i = 0$ for $i > n$. If $\mathcal{N} = \{0, 1, \ldots, |b_0| - 1\}$ we call the NS a **CNS**.

The fundamental domain I

Note that given a number $x \in \mathbb{C}$ we can write $x = x' + r$ where $x' \in \mathbb{Z}[i]$ and r with a digit expansion that has only negative exponents. Given a CNS let

$$\mathcal{F} = \left\{ \sum_{j=1}^{\infty} b^{-j} d_j, d_j \in \mathcal{N} \right\}. \tag{16}$$

We can calculate $\mathcal{F}$ for the following bases:

- $10 : [0, 1]$
- $2 : [0, 1]$
- $-10 : [-\frac{10}{11}, \frac{1}{11}]$
- $-2 : [-\frac{2}{3}, \frac{1}{3}]$
- $-1 - i$:?????

The fundamental domain II

We can identify $\mathbb{C}$ with $\mathbb{R}^2$ via

$$\Phi(a+ib) := \begin{pmatrix} a \\ -b \end{pmatrix} \tag{17}$$

and multiplication with $c+id$ by matrix multiplication from the right with

$$B = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \tag{18}$$

Now if $x \in \mathcal{F}$, then it has a digit expansion $0.d_1 d_2 d_3 \ldots$ in base b with $d_j \in \mathcal{N}$. Multiplication with b yields $bx = d_1.d_2 d_3, \ldots \in \bigcup_{d \in \mathcal{N}} (\mathcal{F} + d)$

The fundamental domain III

Definition

Let $(P(x), \mathcal{N})$ be a NS and Φ, B as described above. The **fundamental domain** $\mathcal{F} \subset \mathbb{R}^n$ of the NS is defined by

$$B\mathcal{F} = \bigcup_{d \in \mathcal{N}} (\mathcal{F} + \Phi(d)) \Leftrightarrow \mathcal{F} = \bigcup_{d \in \mathcal{N}} (B^{-1}\mathcal{F} + B^{-1}\Phi(d)) \tag{19}$$

which is equivalent to

$$\mathcal{F} = \left\{ \sum_{i=1}^{\infty} B^{-i}\Phi(d_i), d_i \in \mathcal{N} \right\}. \tag{20}$$

The base $-i-1$

We recall the definition of the Twin dragon

Definition

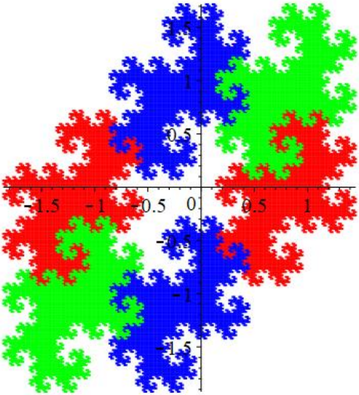
The Twin Dragon $\mathcal{K} \subset \mathbb{R}^2$ is the attractor of the IFS $\{f_1, f_2\}$ with $f_1(x) = B^{-1}x, f_2(x) = B^{-1}(x + (1,0))$ and

$$B = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \tag{21}$$

.

- Matrix corresponds to the complex number $\alpha = -1 - i$ with minimal polynomial $P(x) = x^2 + 2x + 2$
- P forms a CNS:
  Every complex number has a radix representation in base $-1 - i$ and digits 0,1.
- The fundamental domain of this CNS is the Twin Dragon.

Fractal tilings



- The plane can be tiled by the Twin Dragon using the lattice $\mathbb{Z}[i]$.
- The address of a point $x \in \mathcal{K}$ coincides with the digit expansion in base α.
- These algebraic number theoretic properties are used to determine the geometric structure (Hausdorff dimension etc.)

Characterization of the boundary

The boundary of $\mathcal{K}$ is fractal. What characterizes the points in the boundary?
This leads to the notion of **Büchi Automata**.

What are automata?

Directed graphs with vertices called **states**. Further all the arrows are **labeled** over an **alphabet** $\mathcal{A}$. We allow multiple arrows between two states and also different arrows with the same label (**non deterministic**). We call an automaton **finite**, if it has only finitely many states.

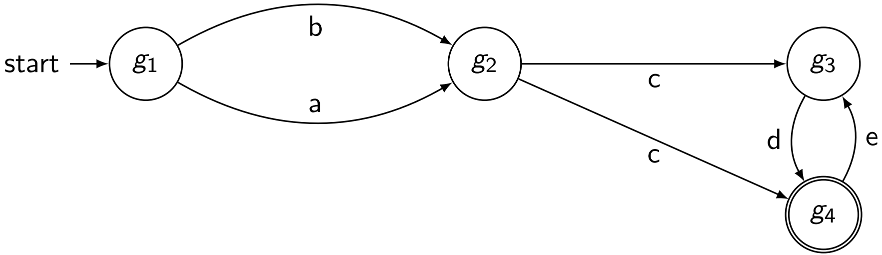


Figure: An automaton accepting $\{ac[de]_\infty, ac[ed]_\infty, bc[de]_\infty, bc[ed]_\infty\}$

Now we can describe paths by the words over $\mathcal{A}$ describing the labels. It is possible to define infinite words, and automata accepting infinite paths. We call these **Büchi automata**.

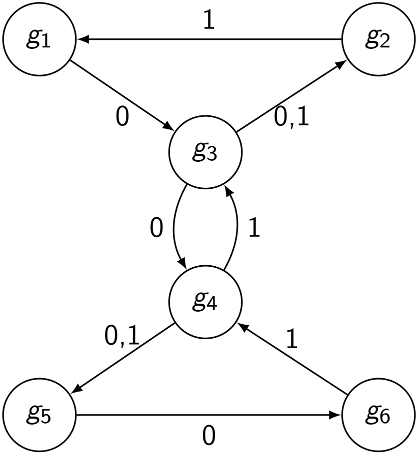


Figure: The automaton $\mathcal{G}$ characterizing $\partial\mathcal{K}$

Intersections with rational lines

The technique proposed in Akiyama and Scheicher, *Intersection two-dimensional fractals with lines* [1], is to find a Büchi automaton to characterize the points in the intersection with a line with rational parameters.

Theorem

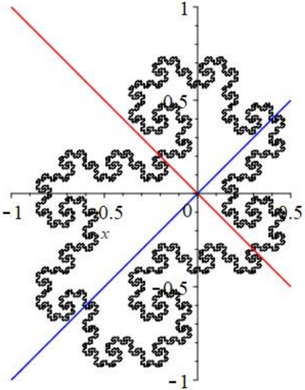
The intersection $\mathcal{K} \cap \{y = 0\}$ *consists of the line segment* $\{(x, 0) : x \in [-\frac{4}{5}, \frac{1}{5}]\}$ *and the intersection* $\partial\mathcal{K} \cap \{y = 0\}$ *consists only of the endpoints of the line segment. The intersection* $\mathcal{K} \cap \{x = 0\}$ *consists of the line segment* $\{(0, y) : y \in [-\frac{2}{5}, \frac{3}{5}]\}$ *and the intersection* $\partial\mathcal{K} \cap \{x = 0\}$ *consists only of the endpoints of the line segment.*

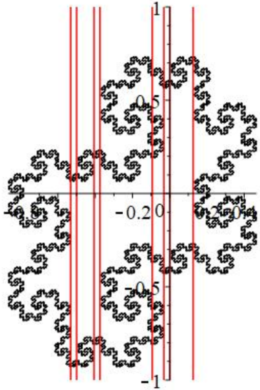
My own results

I investigated the two diagonals and found a similar result. [5]

Theorem

The intersection $\mathcal{K} \cap \Delta$ *consists of the line segment* $\{(x,x) : x \in [-\frac{3}{5}, \frac{2}{5}]\}$*.*
The intersection $\mathcal{K} \cap \overline{\Delta}$ *consits of the line segment* $\{(x,-x) : x \in [-\frac{2}{10}, \frac{3}{10}]\}$*. The intersections with the boundary are only the endpoints.*


1
0.5
-1
-0.5
0
0.5
x
-0.5
-1


1
0.5
-0.2
0
-0.5
-1

Vertical lines and behavior in the limit

- Further I found an infinite class of vertical lines that intersect only in intervals.
- A lemma shows that, for certain values $R \in [-\frac{13}{15}, \frac{7}{15}]$ we can approximate the line $\{x = R\}$ by the lines $\{x = R_N\}$, whose Büchi automata can be easily determined.
- In the limit these Büchi automata "approximate" a infinite automaton, but in some cases it can be represented again as a Büchi automaton.

A somewhat more interesting result

Using this method I could find a line with a more interesting dimension, but still not of Marstrand type. [5]

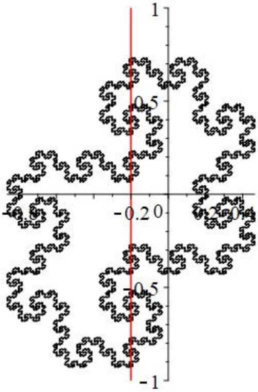
Theorem

$$\mathcal{K} \cap \left\{ x = -\frac{1}{5} \right\} = \left\{ (x, y) \in \mathbb{R}^2 : x = -\frac{1}{5}, y = 0.[d_1 d_2 d_3 d_4 \ldots]_{-4} : d_i \in \{-2, 0, 1, 3\} \right\}$$

The intersection with the boundary $\partial\mathcal{K}$ *are points with* $y = 0.[d_1 d_2 d_3 d_4 \ldots]_{16}$ *with either* $d_i \in \{-14, -12, -8, -6\}$ *for all* i *or* $d_i \in \{-1, 3, 9, 11\}$ *for all* i*. The Hausdorff dimension of* $\mathcal{K} \cap \{x = -\frac{1}{5}\}$ *is 1 and*

$$\dim \left(\partial\mathcal{K} \cap \left\{ x = -\frac{1}{5} \right\} \right) = \frac{\log 3}{\log 4} \approx 0.7925. \qquad (22)$$

Outlook



- It can be proven, that rational lines never intersect the Twin Dragon with Marstrand dimension.
- There already exists a draft of a paper involving Shigeki Akiyama, Benoît Loridant, Wolfgang Steiner and myself, including my results and the proof of this conjecture.

- S. Akiyama and K. Scheicher, *Intersecting two-dimensional fractals with lines*, Acta Scientiarium Mathematicarum, 71 (2005), pp. 555–580.
- P. Duvall, J. Keesling, and A. Vince, *The hausdorff dimension of the boundary of a self-affine tile*, Jounral of the London Mathematical Society, 2 (2000), pp. 748–760.
- G. A. Edgar, *Measure, Topology, and Fractal Geometry (Undergraduate Texts in Mathematics)*, Springer, 2013.
- K. J. Falconer, *The Geometry of Fractal Sets (Cambridge Tracts in Mathematics)*, Cambridge University Press, 1985.
- P. Grosskopf, *Intersecting the twin dragon with rational lines*, Master's thesis, mar 2020.
- M. Yamaguchi, M. Hata, and J. Kigami, *Mathematics of Fractals (Translations of Mathematical Monographs)*, Amer Mathematical Society, 1997.

Thank you for your attention!
I am looking forward to your questions!