Contact structures

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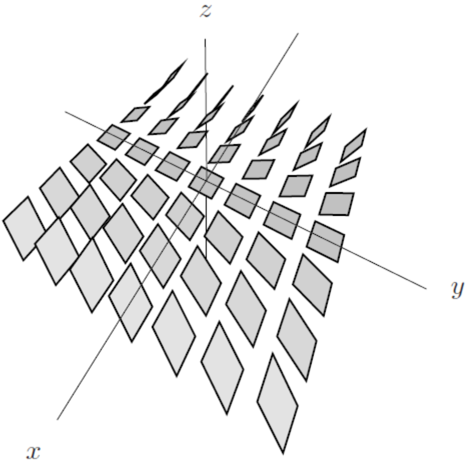
Definition

Definition (Contact structure)

A contact structure on a manifold M of odd dimension $2n+1$ is a hyperplane field ξ which is maximally non-integrable.

Hyperplane field

- A hyperplane field is the association with each point of M of a codimension 1 plane of its tangent space.



Maximally non-integrable

- A hyperplane field is maximally integrable if for each point x of M, there exists a submanifold $N \subset M$ of dimension $2n$ passing through x and such that for all $n \in N$, $T_n N = \xi_n$. The set of N is called a foliation.

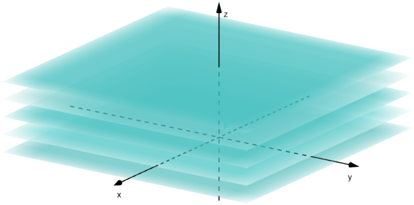


FIGURE – Foliations on $\mathbb{R}^3$ and on the torus $S^1 \times S^1$.

Definition with a differential 1-form

We can see this hyperplane field as the kernel of a differential 1-form, where a 1-form is an application which associate with each point a linear form on its tangent space.

$$\alpha : \mathbb{R}^n \to \mathscr{L}(T\mathbb{R}^n, \mathbb{R})$$

Example

On $\mathbb{R}^3$, (x, y, z),we can consider the tangent space of each point (x, y, z) as $\mathbb{R}^3$, with coordinates $(\partial_x, \partial_y, \partial_z)$.
Let the differential form $\alpha = dz$, which associate with a point (x, y, z) the linear form

$$\alpha_{(x,y,z)} : A = (a_1, a_2, a_3) \mapsto a_3$$

whose kernel is

$$\ker \alpha = \langle \partial_x, \partial_y \rangle \subset T_{(x,y,z)}\mathbb{R}^3$$

How can we define the fact for $\ker\alpha$ to be maximally integrable or not ?

- Frobenius theorem : $\alpha \wedge d\alpha \equiv 0$.
  In the previous example on $\mathbb{R}^3$, we have $\alpha = dz$, so $d\alpha = 0.$;
- Contact condition : $\alpha \wedge (d\alpha)^n \neq 0$.

Definition (Contact structure)

A contact structure on a amanifold M of odd dimension $2n+1$ is a hyperplane field ξ defined as the kernel of a differential 1-form α, satisfying on each point of M,

$$\alpha \wedge (d\alpha)^n \neq 0$$

*The form α is called a **contact form** and the pair (M, ξ) a **contact manifold**.*

Example

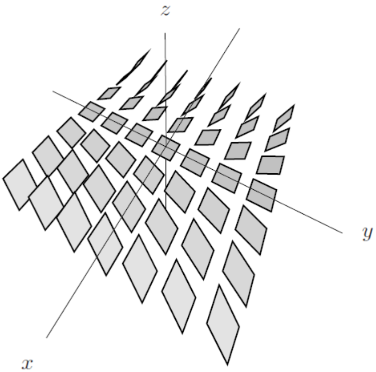


FIGURE – The standard contact structure on $\mathbb{R}^3$ is given by $\alpha = dz - xdy$.

In general, the standard contact structure on $\mathbb{R}^{2n+1}$, with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ is $\alpha = dz - \sum_1^n x_i dy_i$.

Connection with symplectic geometry

- Let α be a contact for on $\mathbb{R}^{2n+1}$. Then the form $d\alpha$ is a symplectic form on the vector space $\ker\alpha$.
  $\implies$ A contact manifold is always of odd dimension.
- Let $(M, \xi = \ker\alpha)$ be a contact manifold. Then the manifold $M \times \mathbb{R}$, equipped with $\omega = d(e^t\alpha)$ is a symplectic manifold, called the *symplectisation* of (M, ξ).

  Likewise, from a symplectic manifold $(W, \omega = d\lambda)$, we can build a contact manifold, taking $W \times \mathbb{R}$ with the 1-form $\alpha = dz - \lambda$. This new space is the *contactisation* of (W, ω).

Connection with symplectic geometry

- We have a Darboux theorem for contact forms : : On all contact manifold (M, α), there exists local coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$ on M such that in these coordinates,

$$\alpha = dz - \sum_{1}^{n} x_i dy_i.$$

Lie contact transformations

We consider $\mathbb{R}^2$ with coordinates (x, z). We call *line element* a line passing through (x, z) of slope p.

$$\{\text{line elements}\} \simeq \left(\mathbb{R}^3, (x, z, p)\right).$$

Equation for a line of slope p $p : dz - pdx = 0$. On the space of line elements, this form defines a contact structure.

Definition

A ***contact transformation*** *is a diffeomorphism of* $\mathbb{R}^3$

$$f : (x, z, p) \mapsto (x_1, z_1, p_1)$$

such that

$$dz_1 - p_1 dx_1 = \rho(x, z, p)(dz - pdx)$$

for a non vanishing function $\rho : \mathbb{R}^3 \to \mathbb{R}$.

Example : Legendre transformation

Example

Let $z : \mathbb{R} \to \mathbb{R}$ be a strictly convex function such that $x \mapsto z'(x)$ is a diffeomorphism of $\mathbb{R}$.
For all $p \in \mathbb{R}$, we define the function Z_p :

$$Z_p(x) = px - z(x).$$

It admit a unique maximum $x = x(p)$ given by

$$\frac{dz}{dx}(x(p)) = p.$$

Let us define a new function z_1 :

$$z_1(p) = Z_p(x(p)) = px(p) - z(x(p)) = \max_{x \in \mathbb{R}} Z_p(x)$$

Example : Legendre transformation

$$\frac{dz_1}{dp}(p) = x(p) + p\frac{dx}{dp}(p) - \frac{dz}{dx}(x(p))\frac{dx}{dp}(p) = x(p).$$

$$\implies dz_1 - x(p)dp = 0$$

The transformation

$$f : (x, z, p) \mapsto (x_1 := p, z_1 := px - z, p_1 := x)$$

is a contact transformation of $\mathbb{R}^3$ (with $\rho \equiv 1$), called the *Legendre transform* of z.

Reeb vector field

Definition

Associated with a contact form α, one has the ***Reeb vector field*** *R_α, uniquely defined by the equations :*

- $d\alpha(R_\alpha, -) \equiv 0$;
- $\alpha(R_\alpha) \equiv 1$.

Example

On $\mathbb{R}^3$ with the standard contact form $\alpha = dz - xdy$, the Reeb vector field is ∂_z.

Contactomorphisms

Definition

Two contact manifolds (M_1, ξ_1) *et* (M_2, ξ_2) *are* ***contactomorphic*** *if there is a difféomorphism* $f : M_1 \to M_2$ *with* $Tf(\xi_1) = \xi_2$*, where* $Tf : TM_1 \to TM_2$ *denotes the differential of* f*.*

If $\xi_i = \ker \alpha_i$*,* $i = 1, 2$*, this is equivalent to saying that* α_1 *and* $f^*\alpha_2$ *give the same hyperplane field, and equivalent to the existence of a non vanishing function* $\lambda : M_1 \to \mathbb{R}\backslash\{0\}$ *such that* $f^*\alpha_2 = \lambda\alpha_1$*.*

We can speak of a ***strict contactomorphism*** *between* ***stricts contact manifolds*** (M_1, α_1) *and* (M_2, α_2) *if* $f^*\alpha_2 = \alpha_1$*.*

Example

- $\mathbb{R}^3$ with its standard contact structure $\alpha = dz - xdy$ is contactomorphic to $\mathbb{R}^3$ with cylindrical coordinates (r, φ, z) and contact structure $\beta = dz + r^2 d\varphi$, by the diffeomorphism

$$f(x, y, z) = \left(\frac{x+y}{2}, \frac{y-x}{2}, z + \frac{xy}{2} \right).$$

- However, $(\mathbb{R}^3, \xi = \ker \alpha)$ is not contactomorphic to $\mathbb{R}^3$ with the 1-form $\alpha_{ot} = \cos(r)dz + r\sin(r)d\varphi$, in cylindrical coordinates (r, φ, z).
  This was proved by D. Bennequin in 1983.
- $(\mathbb{R}^{2n+1}, \xi = \ker \alpha)$ is contactomorphic to $S^{2n+1} \backslash \{p\}$, where p is any point of S^{2n+1}, with its standard contact structure $\alpha_0 = \sum_1^{n+1}(x_i dy_i - y_i dx_i)$.

Theorem (**Gray stability**)

Let ξ_t, $t \in [0,1]$ *be a smooth family of contact structures on a closed manifold* M. *Then there is an isotopy* $(\psi_t)_{t\in[0,1]}$ *of* M *such that*

$$T\psi_t(\xi_0) = \xi_t \text{ pour tout } t \in [0,1]$$

Warning, Gray stability only applies to contact structures, not contact forms !

Lemma

If $f : (M_1, \alpha_1) \to (M_2, \alpha_2)$ *is a strict contactomorphism between two strict contact manifolds, then the Reeb vector fields verify* $Tf(R_1) = R_2$.

Definition

Let (M, ξ) ba a contact manifold. A submanifold L of (M, ξ) is called an ***isotropic submanifold*** *(or* ***integrable****) if $T_pL \subset \xi_p$ for all $p \in L$.*

An isotropic submanifold $L \subset (M^{2n+1}, \xi)$ of maximal dimension n is called a ***Legendrian submanifold****.*

Let (M, ξ) be a contact manifold. A submanifold M' of M with a contact structure ξ' is called a ***contact submanifold*** *of (M, ξ) if $TM' \cap \xi_{|M'} = \xi'$.*

Example

- Thermodynamics (J.W. Gibbs, 1873) .

We consider the following quantities : v the volume, p the pressure, t the temperature, ε the energy, ν the entropy of a given body in any state, W the work done et H the heat received by the body from passing in one state to another. These are subjects to the relations expressed by the following differential equations :

$$d\varepsilon = dH - dW, \; dW = pdv, \; dH = td\nu.$$

Eliminating dW et dH, we have

$$d\varepsilon = td\nu - pdv.$$

This last equations defines a contact structure on the phase space of dimension 5. The set of possible state of a body defines a Legendrian submanifold for this contact form.

Partial differential equations of first order

- We consider the following Cauchy problem :

$$\begin{cases} \Phi(x, f, f') = 0 \\ f(x_0) = u_0 \text{ et } f'(x_0) = p_0 \end{cases}$$

for $f : \mathbb{R}^n \to \mathbb{R}$ and $x_o \in V^{n-1}$, initial data ;

Contact structure on the 1-jets space

We consider

$$J^1(\mathbb{R}^n, \mathbb{R}) = \left\{(x, f(x), f'(x)) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \mid f : \mathbb{R}^n \to \mathbb{R}\right\}$$

manifold of dimension $2n+1$, equipped with the coordinates (x, u, p).

- There is a contact form defined by $\alpha = du - pdx$.
- The **1-jet** of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $x \in \mathbb{R}^n$ is the set

$$J^1_f(x) = \left\{f(x), f'(x)\right\}$$

- The **1-graph** of f is the submanifold formed by 1-jets of f at every point x of $\mathbb{R}^n$:

$$J^1_f = \left\{x, f(x), f'(x) | x \in \mathbb{R}^n\right\}$$

  It is a submanifold of dimension n of $J^1(\mathbb{R}^n, \mathbb{R})$.

Reformulating the equation

In order to solve $\Phi(x, u, p) = 0$, we have, in $J^1(\mathbb{R}^n, \mathbb{R})$:

- A submanifold $E^{2n} = \{(x, u, p) \mid \Phi(x, u, p) = 0\}$;
- A submanifold $N^{n-1} = \left\{(x_0, u_0, p_0) \mid x_0 \in V^{n-1}\right\}$;

and we are looking for a submanifold $Y^n \subset E^{2n}$, containing N^{n-1}, and which is the 1-graph of a function f.

Geometry on a hypersurface in a contact manifold

Let E^{2n} be a smooth hypersurface in a contact manifold (M^{2n+1}, Π).

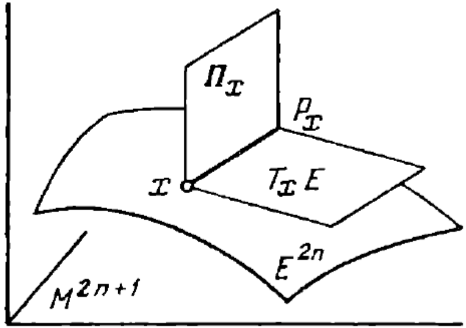
Definition

E^{2n} is said to be **non-characteristic** if its tangent plane $T_x E$ is transverse to Π_x at every point $x \in E$.
In this case, the intersection

$$P_x = T_x E \cap \Pi_x$$

is called the **characteristic plane** at the point x.

Geometry on a hypersurface in a contact manifold

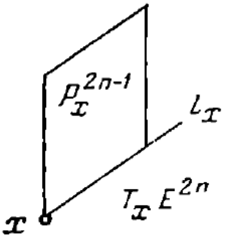


Characteristic direction on a hypersurface

Definition

The **characteristic direction** l_x at a point x on a non characteristic hypersurface E is the orthogonal, in the contact plane Π_x, of the characteristic plane P_x for the symplectic form $d\alpha$.

$$(\Pi_x \ni X) \in l_x \iff \forall\, Y \in T_xE,\ d\alpha(X,Y) = 0$$



The Cauchy problem for the field of characteristic directions

Definition

The **Cauchy problem** for a hypersurface E^{2n} in a contact manifold (M^{2n+1}, Π) with initial manifold N^{n-1} consist of determining an integral manifold Y^n of the field of contact planes, lying in E^{2n}, and containing the initial manifold N^{n-1}.

The Cauchy problem for the field of characteristic directions

Theorem (V. Arnold)

Let $x \in N^{n-1}$ be a non-characteristic point of the initial manifold N^{n-1} ($l_x \notin T_x N$).
There exists a neighbordhood U of x such that the solution of the Cauchy problem for $E \cap U$ with initial condition $N \cap U$ exists and is locally unique.
The manifold Y consists of characteristics passing through the points of the initial manifold N.

Existence : Martinet's construction

Theorem (Martinet, 1971)

Every closed, orientable 3-manifold M admits a contact structure

The proof uses the fact that S^3 admits a contact structure and Lickorish-Wallace Theroem : Every orientable, connected, closed manifold of dimension 3 can be obtain from S^3 by surgery along a collection of embedded circles.

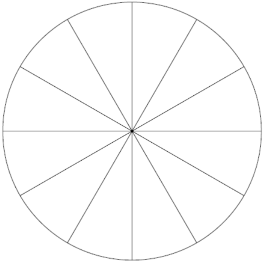
Dichotomy of contact structures (Y. Eliashberg, 1989)

Denote by Δ the disc $\{z = 0, \rho \leq \pi\} \subset \mathbb{R}^3$ with cylindric coordinates (ρ, ϕ, z). The boundary $\partial\Delta$ of this disc is a Legendrian curve for ξ_{ot} defined by

$$\cos(\rho)dz + \rho\sin(\rho)d\phi$$

.

Its characteristic foliation consists of the radius of the disc, and every point of the boundary circle is singular.



Instead we will consider $\Delta = \{z = \varepsilon\rho^2, \rho \leq \pi\}$.
This new disc is close to the original, tangent to ξ_{ot} only at the origin, and has the following foliation.

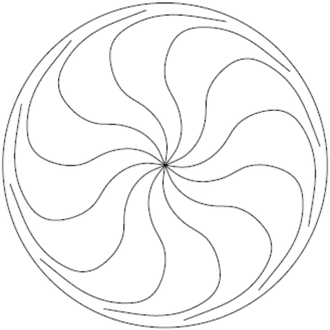


FIGURE – Characteristic foliation of the new Δ .

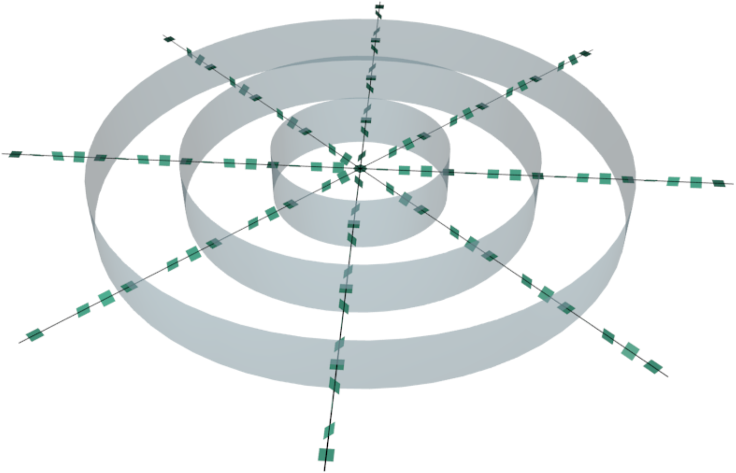
Overtwisted contact structures

Definition

This disc is called the ***standard overtwisted disc****.*
A contact structure on a connected 3-manifold amitting a contact embedding of this disc into M is called ***overtwisted****. Otherwise, it is called* ***tight****.*

Example

The contact structure defined by $\cos(\rho)dz + \rho\sin(\rho)d\varphi$ is overtwisted.



Classification of overtwisted contact structures

Theorem (Y. Eliashberg, 1989)

Let M be a connexe, oriented 3-manifold. Let $Distr(M)$ be te set of plane fields on M et $Cont^{ot}(M)$be the set of overtwisted contact structures. The inclusion

$$j : \mathrm{Cont}^{ot}(M) \to \mathrm{Distr}(M)$$

is a weak homotopy equivalence : it induces a bijection between the arcwise connected components of π_0, et isomorphisms between the higher homotopy groups $\pi_k, k \geq 1$.

Bijection on π_0.

- Injection = Homotopy among plane fields $\implies$ homotopy among contact structures.
- Surjection = Every plane field is homotopic to an overtwisted contact structure.

Classification of overtwisted contact structures in all odd dimension

Definition

An ***almost contact structures*** *on a manifold M is a pair (α, β), where α is a non-vanishing 1-form on M, et β a non-degenerate 2-form on $\xi = \{\alpha = 0\}$.*

Let M be a $(2n+1)$-manifold, $A \subset M$ a closed subspace such that $M \backslash A$ is connected. Let ξ_0 be an almost contact structure on M which is a genuine contact structure on a neighborhood of A.
Denote by $\mathfrak{Cont}_{ot}(M, A, \xi_0)$ the space of contact structures on M, overtwisted on $M \backslash A$ and coincide with ξ_0 on a neighborhood of A, and $\mathfrak{cont}(M, A, \xi_0)$ the space of almost contact structures that agree with ξ_0 on a neighborhood of A.

Classification of overtwisted contact structures in all odd dimension

Theorem (M. Borman, Y. Eliashberg, E. Murphy, 2015)

The inclusion

$$j : \mathfrak{Cont}_{ot}(M, A, \xi_0) \to \mathfrak{cont}(M, A, \xi_0)$$

induces an isomorphism

$$j_* : \pi_0(\mathfrak{Cont}_{ot}(M, A, \xi_0)) \to \pi_0(\mathfrak{cont}(M, A, \xi_0))$$

Classification of tight contact structures

Theorem (Décomposition en somme connexe)

There is a one-to-one correspondence between the set of tight contact structures on a connected sum $M = M_1 \# M_2$ and the product of the sets of tight contact structures on M_1 and M_2.

Theorem (V. Colin, E. Giroux, K.Honda, 2003)

Every 3-manifold has only finitely many homotopy classes of 2-plane fields which carry tight contact structures.

Theorem (Y. Eliashberg)

The standard contact structure $\sum_1^2 x_i dy_i - y_i dx_i$ on S^3 is the unique tight contact structure up to isotopy.

This theorem gives an other proof of the Cerf Theorem

Theorem (Cerf, 1968)

Any diffeomorphism of S^3 extends to D^4.

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