Introduction to toric varieties

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Outline

1. Notions of algebraic geometry
2. Affine toric varieties
3. Abstract toric varieties

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2. Affine toric varieties
3. Abstract toric varieties

Definition

A subset $A \subseteq \mathbb{C}^n$ is an algebraic set if there exists an ideal $I \in \mathbb{C}[X_1, \ldots, X_n]$ such that

$$A = \{x \in \mathbb{C}^n | \forall P \in I, P(x) = 0\}.$$

The topology of $\mathbb{C}^n$ such that closed sets are algebraic sets is **Zariski topology**.

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Ideal of an affine variety and coordinate ring

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To an algebraic variety V we can associate an ideal given by :

$$\mathcal{I}(V) := \{P \in \mathbb{K} | \forall x \in V, P(x) = 0\}.$$

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To an algebraic variety $V \subseteq \mathbb{C}^n$ we can associate a ring, named **coordinate ring**, given by :

$$\mathbb{C}[V] := \mathbb{C}[X_1, ..., X_n]/\mathcal{I}(V).$$

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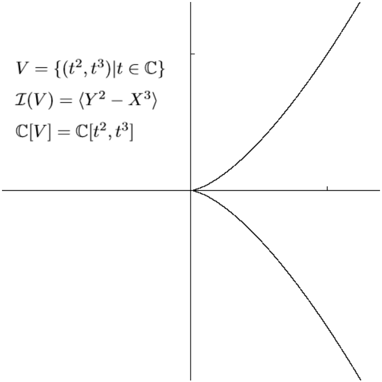
An exemple

Exemple

The cuspidal curve is the curve defined by the equation $Y^2 = X^3$.



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The complex torus

Remark

For a given polynomial $f \in \mathbb{C}[X_1, ..., X_n]$, we can see the open set $U = \mathbb{C}^n \setminus \{x | f(x) = 0\}$ as an algebraic variety of $\mathbb{C}^{n+1}$ corresponding to the ideal $\langle 1 - fY \rangle \subseteq \mathbb{C}[X_1, \ldots, X_n, Y]$.

Definition

The complex **torus** is $(\mathbb{C}^*)^n = \mathbb{C}^n \setminus \mathcal{V}(X_1 \cdots X_n)$ with coordinate ring $\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$.

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Toric varieties

Definition

An algebraic variety V containing a torus as dense open subset and such that the torus acts (algebraically) on the variety is called a **toric variety**.

Character of a torus

Definition

Let T be a torus, a **character** m of T is a group homomorphism $\chi^m : T \to \mathbb{C}^*$.

The set of characters forms a group that we will denote M.

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Character lattice

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All characters are of the form :

$$\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*, (t_1, \ldots, t_n) \mapsto t_1^{a_1} \ldots t_n^{a_n}.$$

Therefore $M \simeq \mathbb{Z}^n$ and we will write $m = (a_1, \ldots, a_n)$. The group $\mathbb{Z}^n$ is the definition of **lattice**. Thus M is a lattice (i.e. a free abelian group of finite rank).

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The combinatorics theorem

Theorem

Let T be a torus and $\mathcal{A} = \{m_1, \ldots, m_s\} \subset M$, consider the map

$$\phi_{\mathcal{A}} : T \to \mathbb{C}^s, t \mapsto (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)).$$

Then the Zariski closure of the image of $\phi_{\mathcal{A}}$ is a toric variety and all toric varieties arise this way.

We will denote the variety generated this way $Y_{\mathcal{A}}$.

Caracterizing the toric ideals

Corollary

Let T be a torus and $\mathcal{A} = \{m_1, \ldots, m_s\} \subset M$, consider the sublattice

$$L = \left\{ u \in \mathbb{Z}^s | \sum_{i=1}^{s} u_i m_i = 0 \right\}.$$

Then $\mathcal{I}(Y_\mathcal{A}) = \langle x^\alpha - x^\beta | \alpha, \beta \in \mathbb{N}^s, \alpha - \beta \in L \rangle$.

Generate a toric variety with a cone

Process

Let us fix $n \in \mathbb{N}^*$, we look at the following steps :

1. We fix a polyhedral cone σ in $\mathbb{R}^n$.
2. We look at its dual $\sigma^\vee$ in $\mathbb{R}^n$.
3. We denote S the set of points of $\sigma^\vee \cap \mathbb{Z}^n$.
4. The variety U_σ is the variety with coordinate ring $\langle x^s | s \in S \rangle \subseteq \mathbb{C}[X_1, ..., X_n]$.

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Main result

Theorem

The variety U_σ is a **normal** toric variety and all normal toric varieties arise this way.

Definition

An algebraic variety V is **normal** if its coordinate ring $\mathbb{K}[V]$ is normal, ie if an element $v \in \mathbb{K}(V)$ is a solution of a monic polynomial of $\mathbb{K}[V][X]$, then $v \in \mathbb{K}[V]$.

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Abstract varieties

Definition

An abstract variety is the collection of a set of affine varieties (V_α) glued together by isomorphisms on open sets.

Example

The projective plane $\mathbb{P}^1(\mathbb{C})$ is the union of $\mathbb{C} = V_1$ and $\mathbb{C} = V_2$ with the gluing $g : V_1 \setminus \{0\} \to V_2 \setminus \{0\}, x \mapsto 1/x$.

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Equivalent of cones for abstract varieties

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Fan

Definition

A **fan** Σ in $\mathbb{R}^n$ is a finite collection of cones in $\mathbb{R}^n$ which is stable by restriction to faces of cones and such that the intersection of two cones of Σ is the greatest common face of those two cones.

This is the good notion !

Theorem

The variety X_Σ generated by the fan Σ is a normal toric variety and all normal toric varieties arise this way.

Orbit-cone correspondence

Theorem

Let Σ be a fan and X_Σ the corresponding toric variety. Let us denote T the torus of X_Σ, then :

1. There is a bijective correspondence
$$\{\text{cones } \sigma \in \Sigma\} \longleftrightarrow \{T-\text{orbits of } X_\Sigma\}.$$
2. The dimension of the orbit corresponding to σ is the codimension of σ.
3. The affine variety U_σ is the union of the orbits of the faces of σ.
4. For two cones $\tau, \sigma \in \Sigma$, τ is a face of σ if and only if the orbit corresponding to σ is a subset of the closure of the orbit corresponding to τ.

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Conclusion

Thank you for your attention !

Feel free to ask any question.

(Bibliography on demand.)