Braided monoidal categories, braces, and the quantum Yang-Baxter equation

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Quantum Yang–Baxter equation

• It first appeared in [C.N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, 1967] and, independently, in [R.J. Baxter, *Partition function of the eight-vertex lattice model*, 1972].

Given a vector space V, a morphism $\sigma : V \otimes V \to V \otimes V$ is a solution of the *quantum Yang–Baxter equation* if the following equality holds:

$$(\sigma \otimes \mathrm{Id}_V)\sigma_{13}(\mathrm{Id}_V \otimes \sigma) = (\mathrm{Id}_V \otimes \sigma)\sigma_{13}(\sigma \otimes \mathrm{Id}_V),$$

where $\sigma_{13} = (\mathrm{Id}_V \otimes \tau_{V,V})(\sigma \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes \tau_{V,V})$ and $\tau_{V,V} : v \otimes w \mapsto w \otimes v$.

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The morphism σ satisfies the quantum Yang–Baxter equation if and only if $\tau_{V,V}\sigma$ satisfies the *braid equation*:

$$(\sigma \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes \sigma)(\sigma \otimes \mathrm{Id}_V) = (\mathrm{Id}_V \otimes \sigma)(\sigma \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes \sigma).$$

A pair (V, σ) where $\sigma : V \otimes V \to V \otimes V$ satisfies the braid equation is called a *braided vector space*.

Monoidal categories

Definition (J. Bénabou, 1963)

A *monoidal category* is a category $\mathcal{M}$ endowed with a distinguished object $\mathbf{1}$, called *unit*, and a functor

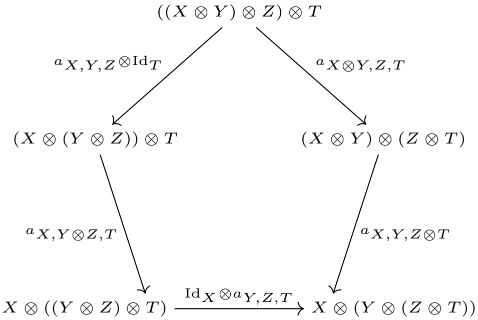
$$\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M},\ (X, Y) \mapsto X \otimes Y,\ (f, g) \mapsto f \otimes g,$$

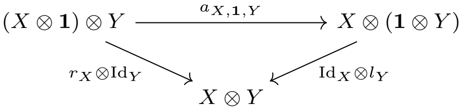
called *tensor product*, and isomorphisms natural in X, Y, Z,

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z),\ l_X : \mathbf{1} \otimes X \to X \text{ and } r_X : X \otimes \mathbf{1} \to X.$$

Here a is called *associativity constraint* while l and r are called left and right *unit constraint*, respectively.

Moreover, the *Pentagon Axiom* and the *Triangle Axiom* are satisfied, i.e. for all X, Y, Z, T in $\mathcal{M}$, the following diagrams commute





A monoidal category will be denoted by $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r)$ or simply by $(\mathcal{M}, \otimes, \mathbf{1})$.

Examples

1. $(\mathsf{Set}, \times, *, a, l, r)$ where $\times$ is the cartesian product, $\{*\}$ is a singleton, and

$$a_{X,Y,Z}(((x,y),z)) = (x,(y,z)), \qquad l_X((*,x)) = x = r_X((x,*)),$$

for all X, Y, Z in Set. Also $(\mathsf{Grp}, \times, \{*\}, a, l, r)$.

2. $(\mathsf{Vec}_{\Bbbk}, \otimes, \Bbbk, a, l, r)$ where $\otimes = \otimes_{\Bbbk}$ and

$$a_{X,Y,Z}((x \otimes y) \otimes z) = x \otimes (y \otimes z), \quad l_X(k \otimes x) = kx = r_X(x \otimes k),$$

for all X, Y, Z in $\mathsf{Vec}_{\Bbbk}$.

3. $(\mathrm{End}(\mathcal{C}), \circ, \mathbf{1}_{\mathcal{C}})$, where $\mathrm{End}(\mathcal{C})$ is the category of the endofunctors of a small category $\mathcal{C}$, $\circ$ is the composition of functors and $\mathbf{1}_{\mathcal{C}}$ is the identity functor. Here a, l and r are the identities, so we have a *strict* monoidal category.

• By Mac Lane's *coherence theorem*, every monoidal category is monoidally equivalent to a strict monoidal category.

Definition

Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a monoidal category. A *monoid* (or *algebra*) in $\mathcal{M}$ is a triple (A, m, u) where A is an object in $\mathcal{M}$ and $m : A \otimes A \to A$ (*multiplication*) and $u : \mathbf{1} \to A$ (*unit*) are morphisms in $\mathcal{M}$ such that the following diagrams commute

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{m\otimes \mathrm{Id}} & A\otimes A \\ {\scriptstyle \mathrm{Id}\otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A\otimes A & \xrightarrow[m]{} & A \end{array} \qquad \begin{array}{ccccc} \mathbf{1}\otimes A & \xrightarrow{u\otimes \mathrm{Id}_A} & A\otimes A & \xleftarrow{\mathrm{Id}_A\otimes u} & A\otimes \mathbf{1} \\ & {\scriptstyle l_A}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle r_A} & \\ & & A & & \end{array}$$

Let (A, m, u) and (A', m', u') be algebras in $\mathcal{M}$. A morphism $f : A \to A'$ in $\mathcal{M}$ is called a *monoid morphism* if

$$\begin{array}{ccc} A\otimes A & \xrightarrow{f\otimes f} & A'\otimes A' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ A & \xrightarrow[f]{} & A' \end{array} \qquad \begin{array}{ccc} & \mathbf{1} & \\ {\scriptstyle u}\swarrow & & \searrow{\scriptstyle u'} \\ A & \xrightarrow[f]{} & A' \end{array}$$

We denote the category by $\mathsf{Mon}(\mathcal{M}, \otimes, \mathbf{1})$ or $\mathsf{Mon}(\mathcal{M})$.

Example

$\mathsf{Mon}(\mathsf{Vec}, \otimes_{\Bbbk}, \Bbbk) = \mathsf{Alg}_{\Bbbk}$, the category of $\Bbbk$-algebras. Objects are $\Bbbk$-vector spaces A with linear maps $m : A \otimes A \to A$ and $u : \Bbbk \to A$ such that

$$(ab)c = a(bc), \qquad 1_A a = a = a 1_A, \qquad \text{for all } a, b, c \in A.$$

Morphisms are $\Bbbk$-linear maps $f : A \to A'$ such that $f(ab) = f(a)f(b)$ and $f(1_A) = 1_{A'}$.

Given A, B in $\mathsf{Alg}_{\Bbbk}$, $A \otimes_{\Bbbk} B$ is in $\mathsf{Alg}_{\Bbbk}$ with $(m_A \otimes m_B)(\mathrm{Id}_A \otimes \tau_{B,A} \otimes \mathrm{Id}_B)$, i.e.

$$(a \otimes b)(a' \otimes b') := aa' \otimes bb', \qquad \text{for all } a, a' \in A, b, b' \in B$$

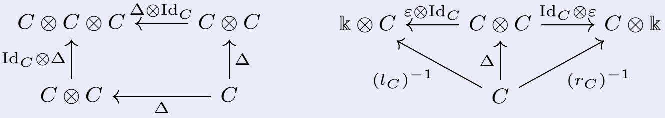
where $\tau_{B,A}(b \otimes a) = a \otimes b$, and unit $1_{A \otimes B} := 1_A \otimes 1_B$.

Reversing all arrows, we get the definitions of *comonoid* and *comonoid morphism*. The corresponding category is denoted by $\mathsf{Comon}(\mathcal{M}, \otimes, \mathbf{1})$ or $\mathsf{Comon}(\mathcal{M})$.

- $\mathsf{Comon}(\mathsf{Vec}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk) = \mathsf{Coalg}_{\Bbbk}$, the category of $\Bbbk$-coalgebras.

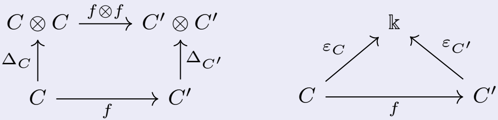
Definition

A *coalgebra* is a $\Bbbk$-vector space C endowed with two $\Bbbk$-linear maps $\Delta : C \to C \otimes C$ (*comultiplication*) and $\varepsilon : C \to \Bbbk$ (*counit*) such that



Usually one uses the *Sweedler's notation* for the comultiplication: $\Delta(x) = x_1 \otimes x_2$. The previous diagrams: $x_1 \otimes x_{2_1} \otimes x_{2_2} = x_{1_1} \otimes x_{1_2} \otimes x_2$, $\varepsilon(x_1)x_2 = x = x_1\varepsilon(x_2)$.

A coalgebra morphism is a linear map $f : C \to C'$ such that



Using the Sweedler's notation: $f(x)_1 \otimes f(x)_2 = f(x_1) \otimes f(x_2)$ and $\varepsilon(f(x)) = \varepsilon(x)$.

Example

1. Let $S \neq \emptyset$ be an arbitrary set and consider the free vector space $\Bbbk S$ with basis S. Then $\Bbbk S$ becomes a coalgebra with comultiplication and counit defined on a generator $s \in S$ by $\Delta(s) = s \otimes s$ and $\varepsilon(s) = 1_\Bbbk$ (s is called *group-like*).
2. Consider the polynomial ring $\Bbbk[X]$ in the variable X. This is a coalgebra by setting $\Delta(X^n) = X^n \otimes X^n$ and $\varepsilon(X^n) = 1_\Bbbk$. Moreover, it is also a coalgebra by setting $\Delta(X^n) = \sum_{i=0}^n X^i \otimes X^{n-i}$ and $\varepsilon(X^n) = \delta_{n,0}$ and these two coalgebra structures are not isomorphic.

Given two $\Bbbk$-coalgebras C and D, then $C \otimes_\Bbbk D$ is a coalgebra via

$$\Delta_{C\otimes D} := (\mathrm{Id}_C \otimes \tau_{C,D} \otimes \mathrm{Id}_D)(\Delta_C \otimes \Delta_D), \quad x \otimes y \mapsto x_1 \otimes y_1 \otimes x_2 \otimes y_2$$

and $\varepsilon_{C\otimes D}(c \otimes d) = \varepsilon_C(c)\varepsilon_D(d)$. A coalgebra C is *cocommutative* if $\Delta_C = \tau_{C,C}\Delta_C$, i.e. $x_1 \otimes x_2 = x_2 \otimes x_1$.

Example

1. Let $S \neq \emptyset$ be an arbitrary set and consider the free vector space $\Bbbk S$ with basis S. Then $\Bbbk S$ becomes a coalgebra with comultiplication and counit defined on a generator $s \in S$ by $\Delta(s) = s \otimes s$ and $\varepsilon(s) = 1_\Bbbk$ (s is called *group-like*).
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Given two $\Bbbk$-coalgebras C and D, then $C \otimes_\Bbbk D$ is a coalgebra via

$$\Delta_{C\otimes D} := (\mathrm{Id}_C \otimes \tau_{C,D} \otimes \mathrm{Id}_D)(\Delta_C \otimes \Delta_D), \quad x \otimes y \mapsto x_1 \otimes y_1 \otimes x_2 \otimes y_2$$

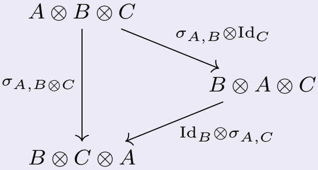
and $\varepsilon_{C\otimes D}(c \otimes d) = \varepsilon_C(c)\varepsilon_D(d)$. A coalgebra C is *cocommutative* if $\Delta_C = \tau_{C,C}\Delta_C$, i.e. $x_1 \otimes x_2 = x_2 \otimes x_1$.

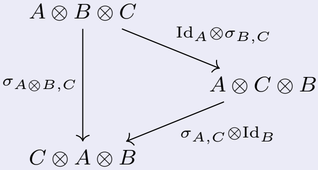
- τ is a *braiding* for the category of vector spaces.

Braided monoidal categories

Definition (A. Joyal, R. Street, 1986)

A monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ is *braided* if, for all X, Y in $\mathcal{M}$, there is an isomorphism $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ (*braiding*), natural in X and Y, such that:





If $\sigma^{-1}_{X,Y} = \sigma_{Y,X}$ for all X, Y in $\mathcal{M}$, then it is said *symmetric monoidal*.
We denote a braided monoidal category by $(\mathcal{M}, \otimes, \mathbf{1}, \sigma)$.

Examples (Symmetric monoidal categories)

1. $(\mathsf{Set}, \times, *, \tau)$ where $\tau_{X,Y} : (x, y) \mapsto (y, x)$ for any X, Y in Set.
2. $(\mathsf{Vec}_{\mathbb{k}}, \otimes_{\mathbb{k}}, \mathbb{k}, \tau)$ where $\tau_{X,Y} : x \otimes y \mapsto y \otimes x$, for any X, Y in $\mathsf{Vec}_{\mathbb{k}}$.

Examples (Symmetric monoidal categories)

1. $(\mathsf{Set}, \times, *, \tau)$ where $\tau_{X,Y} : (x, y) \mapsto (y, x)$ for any X, Y in Set.
2. $(\mathsf{Vec}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk, \tau)$ where $\tau_{X,Y} : x \otimes y \mapsto y \otimes x$, for any X, Y in $\mathsf{Vec}_{\Bbbk}$.

- Given $(\mathcal{M}, \otimes, \mathbf{1}, \sigma)$ and M, N, O in $\mathcal{M}$, the *braid equation* holds:

$$(\sigma_{N,O} \otimes \mathrm{Id}_M)(\mathrm{Id}_N \otimes \sigma_{M,O})(\sigma_{M,N} \otimes \mathrm{Id}_O) = (\mathrm{Id}_O \otimes \sigma_{M,N})(\sigma_{M,O} \otimes \mathrm{Id}_N)(\mathrm{Id}_M \otimes \sigma_{N,O})$$

If there is a forgetful functor $U : \mathcal{M} \to \mathsf{Vec}_{\Bbbk}$, $(M, \sigma_{M,M})$ is a braided vector space for any $M \in \mathcal{M}$.

Bimonoids and Hopf monoids

Definition

Let $(\mathcal{M}, \otimes, \mathbf{1}, \sigma)$ be a braided monoidal category. A *bimonoid* in $\mathcal{M}$ is a datum $(B, m, u, \Delta, \varepsilon)$ where:

1) (B, m, u) is in $\mathsf{Mon}(\mathcal{M})$,
2) (B, Δ, ε) is in $\mathsf{Comon}(\mathcal{M})$,
3) Δ, ε are in $\mathsf{Mon}(\mathcal{M})$ (equivalently, m and u are in $\mathsf{Comon}(\mathcal{M})$):

$$\Delta m = (m \otimes m)(\mathrm{Id} \otimes \sigma_{B,B} \otimes \mathrm{Id})(\Delta \otimes \Delta), \ \Delta u = u \otimes u, \ \varepsilon m = \varepsilon \otimes \varepsilon, \ \varepsilon u = \mathrm{Id}_{\mathbf{1}}.$$

A morphism of bimonoids is a morphism of monoids and comonoids. We denote the category by $\mathsf{Bimon}(\mathcal{M})$.

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Given H in $\mathsf{Bimon}(\mathcal{M})$, the set $\mathrm{Hom}(H, H)$ is a monoid with the *convolution product*: given $f, g : H \to H$ in $\mathcal{M}$, set $f * g := m_H(f \otimes g)\Delta_H$. The unit is $u_H \varepsilon_H$.

Definition

A *Hopf monoid* H in $\mathcal{M}$ is a bimonoid in $\mathcal{M}$ with a morphism $S : H \to H$ (called *antipode*) which is a convolution inverse of Id_H.
A morphism of Hopf monoids is a morphism of bimonoids. We denote the category by $\mathsf{Hopf}(\mathcal{M})$.

Examples:

- $\mathsf{Hopf}(\mathsf{Set}) = \mathsf{Grp}$,
- $\mathsf{Hopf}(\mathsf{Vec}_{\Bbbk}) = \mathsf{Hopf}_{\Bbbk}$, the category of $\Bbbk$-Hopf algebras.

  A $\Bbbk$-*Hopf algebra* is a datum $(H, m, u, \Delta, \varepsilon, S)$, where (H, m, u) is a $\Bbbk$-algebra, (H, Δ, ε) is a $\Bbbk$-coalgebra, Δ and ε are algebra maps:

  $$\Delta(ab) = a_1 b_1 \otimes a_2 b_2, \quad \Delta(1_H) = 1_H \otimes 1_H, \quad \varepsilon(ab) = \varepsilon(a)\varepsilon(b), \quad \epsilon(1_H) = 1_{\Bbbk}$$

  and $S : H \to H$ is a $\Bbbk$-linear map such that $S(b_1)b_2 = \varepsilon(b)1_H = b_1 S(b_2)$.

Modules and comodules in a monoidal category

Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a monoidal category and A an object in $\mathsf{Mon}(\mathcal{M})$.

Definition

A (left) A-module is an object M in $\mathcal{M}$ equipped with a morphism $\mu : A \otimes M \to M$ in $\mathcal{M}$ (called *action*) such that the following diagrams commute.

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{m \otimes \mathrm{Id}} & A \otimes M \\ {\scriptstyle \mathrm{Id} \otimes \mu} \downarrow & & \downarrow {\scriptstyle \mu} \\ A \otimes M & \xrightarrow[\mu]{} & M \end{array} \qquad \begin{array}{ccc} \mathbf{1} \otimes M & \xrightarrow{u \otimes \mathrm{Id}} & A \otimes M \\ & {\scriptstyle l_M} \searrow & \downarrow {\scriptstyle \mu} \\ & & M \end{array}$$

A morphism $f : M \to N$ of left A-modules is a morphism in $\mathcal{M}$ such that $\mu_N(\mathrm{Id} \otimes f) = f\mu_M$. We denote the category by ${}_A\mathcal{M}$.

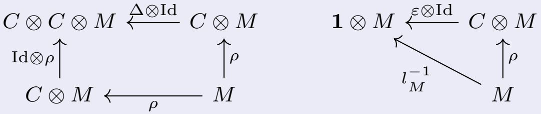
If $\mathcal{M} = \mathsf{Vec}_{\Bbbk}$ one recovers the category of (left) A-modules ${}_A\mathfrak{M}$:

- Objects: $\Bbbk$-vector spaces M endowed with a $\Bbbk$-linear map $\rightharpoonup : A \otimes M \to M$ such that $a \rightharpoonup (b \rightharpoonup m) = ab \rightharpoonup m$ and $1_A \rightharpoonup m = m$
- Morphisms: $\Bbbk$-linear maps $f : M \to N$ such that $f(a \rightharpoonup m) = a \rightharpoonup f(m)$.

Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a monoidal category and C an object in $\mathsf{Comon}(\mathcal{M})$.

Definition

A left C-comodule is an object M in $\mathcal{M}$ endowed with a morphism $\rho : M \to C \otimes M$ in $\mathcal{M}$ (called *coaction*) such that the following diagrams commute.



A morphism $f : M \to N$ of left C-comodules is a morphism in $\mathcal{M}$ such that $\rho_N f = (\mathrm{Id} \otimes f)\rho_M$. We denote the category by ${}^C\mathcal{M}$.

If $\mathcal{M} = \mathsf{Vec}_{\Bbbk}$ one recovers the category of (left) C-comodules ${}^C\mathfrak{M}$:

- Objects: vector spaces M with a linear map $\rho : M \to C \otimes M$, $m \mapsto m_{-1} \otimes m_0$ such that $m_{-1} \otimes {m_0}_{-1} \otimes {m_0}_0 = {m_{-1}}_1 \otimes {m_{-1}}_2 \otimes m_0$ and $\varepsilon(m_{-1})m_0 = m$.
- Morphisms: linear maps $f : M \to N$ s.t. $f(m)_{-1} \otimes f(m)_0 = m_{-1} \otimes f(m_0)$.

If H is in $\mathsf{Bimon}(\mathcal{M})$, the categories $({}_H\mathcal{M}, \otimes, \mathbf{1})$ and $({}^H\mathcal{M}, \otimes, \mathbf{1})$ are monoidal.

1. The constraints a, l, r are the same of $\mathcal{M}$.
2. Given (M, μ_M) and (N, μ_N) in ${}_H\mathcal{M}$, $M \otimes N$ is in ${}_H\mathcal{M}$ with (diagonal) action

$$(\mu_M \otimes \mu_N)(\mathrm{Id}_H \otimes \sigma_{H,M} \otimes \mathrm{Id}_N)(\Delta_H \otimes \mathrm{Id}_{M\otimes N})$$

   If $\mathcal{M} = \mathsf{Vec}_{\Bbbk}$ then $a \rightharpoonup (m \otimes n) := (a_1 \rightharpoonup_M m) \otimes (a_2 \rightharpoonup_N n)$.

   Given M, N in ${}^H\mathcal{M}$, then $M \otimes N$ is in ${}^H\mathcal{M}$ with (diagonal) coaction

$$(m_H \otimes \mathrm{Id}_{M\otimes N})(\mathrm{Id}_H \otimes \sigma_{M,H} \otimes \mathrm{Id}_N)(\rho_M \otimes \rho_N)$$

   If $\mathcal{M} = \mathsf{Vec}_{\Bbbk}$ then $m \otimes n \mapsto m_{-1}n_{-1} \otimes m_0 \otimes n_0$.
3. The unit object $\mathbf{1}$ is a left H-module and a left H-comodule with (trivial) action $\varepsilon_H \otimes \mathrm{Id}_{\mathbf{1}}$ and (trivial) coaction $u_H \otimes \mathrm{Id}_{\mathbf{1}}$.

   If $\mathcal{M} = \mathsf{Vec}_{\Bbbk}$ then $a \rightharpoonup k := \varepsilon(a)k$ and $1_{\Bbbk} \mapsto 1_H \otimes 1_{\Bbbk}$.

- Given a bialgebra H, the categories $({}_H\mathfrak{M}, \otimes_{\Bbbk}, \Bbbk)$ and $({}^H\mathfrak{M}, \otimes_{\Bbbk}, \Bbbk)$ are monoidal.

  Question: Given a bialgebra H, are the latter monoidal categories braided?

Quasitriangular bialgebras

Definition (Drinfeld 1990)

A bialgebra H is said to be *quasitriangular* if there is an invertible element $\mathcal{R} = \mathcal{R}^i \otimes \mathcal{R}_i \in H \otimes H$, called universal $\mathcal{R}$-matrix, such that:

$$\Delta^{\mathrm{op}}(\cdot)\mathcal{R} = \mathcal{R}\Delta(\cdot) \tag{1}$$

$$(\mathrm{Id}_H \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12} \tag{2}$$

$$(\Delta \otimes \mathrm{Id}_H)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{3}$$

where $\mathcal{R}_{12} = \mathcal{R}^i \otimes \mathcal{R}_i \otimes 1_H$, $\mathcal{R}_{13} = \mathcal{R}^i \otimes 1_H \otimes \mathcal{R}_i$ and $\mathcal{R}_{23} = 1_H \otimes \mathcal{R}^i \otimes \mathcal{R}_i$. If in addition $\mathcal{R}^{-1} = \mathcal{R}^{\mathrm{op}}$, then $(H, \mathcal{R})$ is called *triangular*.

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Theorem (Drinfeld 1990)

A bialgebra H is quasitriangular if and only if $({}_H\mathfrak{M}, \otimes_{\Bbbk}, \Bbbk)$ is braided with

$$\sigma^{\mathcal{R}}_{M,N} : M \otimes N \to N \otimes M, \;\; m \otimes n \mapsto (\mathcal{R}_i \cdot n) \otimes (\mathcal{R}^i \cdot m)$$

for all M, N in ${}_H\mathfrak{M}$. Moreover, H is triangular if and only if ${}_H\mathfrak{M}$ is symmetric.

Coquasitriangular bialgebras

Definition

A bialgebra H is said to be *coquasitriangular* if there exists a linear map $\mathcal{R} : H \otimes H \to \Bbbk$, called universal $\mathcal{R}$-form, which is convolution invertible, i.e. there exists $\mathcal{R}^{-1} : H \otimes H \to \Bbbk$ such that

$$\mathcal{R}(x_1 \otimes y_1)\mathcal{R}^{-1}(x_2 \otimes y_2) = \varepsilon(x)\varepsilon(y) = \mathcal{R}^{-1}(x_1 \otimes y_1)\mathcal{R}(x_2 \otimes y_2)$$

such that:

1) $\mathcal{R}(a_1 \otimes b_1)a_2 b_2 = b_1 a_1 \mathcal{R}(a_2 \otimes b_2)$,
2) $\mathcal{R}(a \otimes bc) = \mathcal{R}(a_1 \otimes c)\mathcal{R}(a_2 \otimes b)$,
3) $\mathcal{R}(ab \otimes c) = \mathcal{R}(a \otimes c_1)\mathcal{R}(b \otimes c_2)$.

Moreover, $\mathcal{R}$ is *cotriangular* if $\mathcal{R}^{-1} = \mathcal{R}^{\mathrm{op}}$.

Theorem

A bialgebra H is coquasitriangular if and only if $(\mathfrak{M}^H, \otimes_{\Bbbk}, \Bbbk)$ is braided with

$$\sigma^{\mathcal{R}}_{M,N} : M \otimes N \to N \otimes M, \;\; m \otimes n \mapsto n_0 \otimes m_0 \mathcal{R}(m_1 \otimes n_1)$$

for all M, N in $\mathfrak{M}^H$. Moreover, H is cotriangular if and only if $\mathfrak{M}^H$ is symmetric.

Yetter–Drinfeld modules

Let H be a bialgebra. The category ${}^H_H\mathcal{YD}$ of (left) *Yetter–Drinfeld modules* is defined in the following way:

1) Objects are data $(V, \rightharpoonup, \rho)$, where $(V, \rightharpoonup : H \otimes V \to V)$ is a left H-module, $(V, \rho : V \to H \otimes V, v \mapsto v_{-1} \otimes v_0)$ is a left H-comodule and the following compatibility condition holds true:

$$(h_1 \rightharpoonup v)_{-1} h_2 \otimes (h_1 \rightharpoonup v)_0 = h_1 v_{-1} \otimes (h_2 \rightharpoonup v_0).$$

2) Morphisms are left H-linear maps which are also left H-colinear.

- $({}^H_H\mathcal{YD}, \otimes, \Bbbk)$ is a monoidal category. Given X, Y in ${}^H_H\mathcal{YD}$, $X \otimes Y$ is in ${}^H_H\mathcal{YD}$ with action and coaction

$$h \rightharpoonup_{\otimes} (x \otimes y) := (h_1 \rightharpoonup_X x) \otimes (h_2 \rightharpoonup_Y y), \quad \rho_{\otimes}(x \otimes y) := x_{-1}y_{-1} \otimes x_0 \otimes y_0$$

  and $\Bbbk$ is in ${}^H_H\mathcal{YD}$ with $h \rightharpoonup k := \varepsilon(h)k$ and $\rho(k) = 1_H \otimes k$.

- $({}^H_H\mathcal{YD}, \otimes, \Bbbk, \sigma)$ is a (pre-)braided monoidal category with

$$\sigma_{X,Y} : X \otimes Y \to Y \otimes X, \; x \otimes y \mapsto (x_{-1} \rightharpoonup_Y y) \otimes x_0$$

  We can define $\mathsf{Hopf}({}^H_H\mathcal{YD})$. If H is a Hopf algebra with bijective antipode, $\sigma_{X,Y}$ is bijective.

Yetter–Drinfeld braces and matched pairs of actions

Definition (D. Ferri, A.S.)

A **Yetter–Drinfeld brace** $(H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T)$ consists of

1) A Hopf algebra $H^{\bullet} = (H, \bullet, 1, \Delta, \varepsilon, T)$,
2) An object $H^{\cdot} = (H, \cdot, 1, \Delta, \varepsilon, S)$ in $\mathsf{Hopf}({}^{H^\bullet}_{H^\bullet}\mathcal{YD})$ with action and coaction

$$a \rightharpoonup b := S(a_1) \cdot (a_2 \bullet b), \qquad \mathrm{Ad}_L : a \mapsto a_1 \bullet T(a_3) \otimes a_2$$

such that the following equalities are satisfied:

$$a \bullet (b \cdot c) = (a_1 \bullet b) \cdot S(a_2) \cdot (a_3 \bullet c) \tag{4}$$

$$(a_1 \rightharpoonup b_1) \otimes T(a_2 \rightharpoonup b_2) \bullet a_3 \bullet b_3 = (a_3 \rightharpoonup b_3) \otimes T(a_1 \rightharpoonup b_1) \bullet a_2 \bullet b_2.$$

A morphism of Yetter–Drinfeld braces $f : (H, \cdot, \bullet) \to (K, \cdot, \bullet)$ is a morphism of coalgebras and algebras with respect to $\cdot$ and $\bullet$. We denote this category by $\mathcal{YD}\mathsf{Br}$.

• Under the assumption of cocommutativity, $H^{\bullet}$ and $H^{\cdot}$ are Hopf algebras such that (4) holds, so $(H, \cdot, \bullet)$ is a *cocommutative Hopf brace* [I. Angiono, C. Galindo, L. Vendramin, *Hopf braces and Yang–Baxter operators*, 2017].

Question: Do Yetter–Drinfeld braces produce solutions of the quantum Yang–Baxter equation?

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Definition

Let $(H, \bullet, 1, \Delta, \varepsilon, T)$ be a Hopf algebra. A **matched pair of actions** $(H, \rightharpoonup, \leftharpoonup)$ is given by a pair of actions $\rightharpoonup, \leftharpoonup : H \otimes H \to H$ which are coalgebra morphisms s.t.

$$a \bullet b = (a_1 \rightharpoonup b_1) \bullet (a_2 \leftharpoonup b_2). \tag{5}$$

A morphism of matched pairs of actions $f : (H, \rightharpoonup_H, \leftharpoonup_H) \to (K, \rightharpoonup_K, \leftharpoonup_K)$ is a morphism of Hopf algebras $f : H \to K$ such that:

$$f(a \rightharpoonup_H b) = f(a) \rightharpoonup_K f(b), \quad f(a \leftharpoonup_H b) = f(a) \leftharpoonup_K f(b).$$

The category of matched pairs of actions will be denoted by MP.

Any matched pair of actions $(H, \rightharpoonup, \leftharpoonup)$ satisfies

1) $a \rightharpoonup (b \bullet c) = (a_1 \rightharpoonup b_1) \bullet ((a_2 \leftharpoonup b_2) \rightharpoonup c)$,
2) $(a \bullet b) \leftharpoonup c = (a \leftharpoonup (b_1 \rightharpoonup c_1)) \bullet (b_2 \leftharpoonup c_2)$,
3) $(a_1 \rightharpoonup b_1) \otimes (a_2 \leftharpoonup b_2) = (a_2 \rightharpoonup b_2) \otimes (a_1 \leftharpoonup b_1)$,

so it is a matched pair of Hopf algebras (H, H) in the sense of [S. Majid, 1990].

More precisely, $H \otimes H$ is a Hopf algebra with the tensor coalgebra structure and algebra structure defined by

$$(a \otimes h)(b \otimes g) = a \bullet (h_1 \rightharpoonup b_1) \otimes (h_2 \leftharpoonup b_2) \bullet g, \qquad 1 = 1_H \otimes 1_H.$$

It is called *double cross product* and is denoted by $H \bowtie H$.

Matched pairs of actions are such that $m_\bullet : H \bowtie H \to H$ is a morphism of algebras.

Theorem (D. Ferri, A.S.)

The following functors provide an isomorphism of categories:

- $F : \mathsf{MP} \to \mathcal{YD}\mathsf{Br}$, $((H, \bullet, 1, \Delta, \varepsilon, T), \rightharpoonup, \leftharpoonup) \mapsto (H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T)$ *where*

$$a \cdot b := a_1 \bullet (T(a_2) \rightharpoonup b), \qquad S(a) := a_1 \rightharpoonup T(a_2).$$

- $G : \mathcal{YD}\mathsf{Br} \to \mathsf{MP}$, $(H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T) \to ((H, \bullet, 1, \Delta, \varepsilon, T), \rightharpoonup, \leftharpoonup)$ *where*

$$a \rightharpoonup b := S(a_1) \cdot (a_2 \bullet b), \qquad a \leftharpoonup b := T(a_1 \rightharpoonup b_1) \bullet a_2 \bullet b_2.$$

Under the assumption of cocommutativity, we recover the isomorphism between matched pairs of actions on cocommutative Hopf algebras and cocommutative Hopf braces [I. Angiono, C. Galindo, L. Vendramin, 2017].

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• Yetter–Drinfeld braces (equivalently, matched pairs of actions) produce solutions to the braid equation $\sigma : x \otimes y \mapsto (x_1 \rightharpoonup y_1) \otimes (x_2 \leftharpoonup y_2)$ [J.A. Guccione, J.J. Guccione, C. Valqui, 2024], as cocommutative Hopf braces do.

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• σ is involutive if and only if $\cdot$ is braided commutative, i.e. $m_\cdot = m_\cdot \sigma^{\mathcal{YD}}_{H,H}$ [Y. Li, *Matched pairs and Yang–Baxter operators*, 2025].

Coquasitriangular Hopf algebras and Yetter–Drinfeld braces

Let $(H, \mathcal{R} : H \otimes H \to \Bbbk)$ be a coquasitriangular Hopf algebra and consider the braided vector space $(H, \sigma_{H,H})$, where

$$\sigma_{H,H} : H \otimes H \to H \otimes H,\ a \otimes b \mapsto \mathcal{R}^{-1}(a_1 \otimes b_1)b_2 \otimes a_2\mathcal{R}(a_3 \otimes b_3).$$

One obtains a matched pair of actions $(\rightharpoonup, \leftharpoonup)$ on H in the following way:

$$a \rightharpoonup b := (\mathrm{Id} \otimes \varepsilon)\sigma_{H,H}(a \otimes b) = \mathcal{R}^{-1}(a_1 \otimes b_1)b_2\mathcal{R}(a_2 \otimes b_3),$$

$$a \leftharpoonup b := (\varepsilon \otimes \mathrm{Id})\sigma_{H,H}(a \otimes b) = \mathcal{R}^{-1}(a_1 \otimes b_1)a_2\mathcal{R}(a_3 \otimes b_2),$$

so that $\sigma_{H,H}(a \otimes b) = (a_1 \rightharpoonup b_1) \otimes (a_2 \leftharpoonup b_2)$.

Theorem (D. Ferri, A.S.)

Let $(H, \bullet, 1, \Delta, \varepsilon, T, \mathcal{R})$ be a coquasitriangular Hopf algebra. Then, $(H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T)$ is a Yetter–Drinfeld brace where:

$$a \cdot b = \mathcal{R}^{-1}(a_1 \bullet T(a_3) \otimes b_1) b_2 \bullet a_2, \qquad S(a) = T(a_3)\mathcal{R}(a_1 \otimes T(a_2) \bullet a_4)$$

and the action is given by $a \rightharpoonup b := b_2 \mathcal{R}^{-1}(a \otimes b_1 \bullet T(b_3))$.

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- The object $(H, \cdot, 1, \Delta, \varepsilon, S)$ in $\mathsf{Hopf}({}^{H^\bullet}_{H^\bullet}\mathcal{YD})$ coincides with the **transmutation** of the Hopf algebra $(H, \bullet, 1, \Delta, \varepsilon, T)$ in the sense of [S. Majid, *Transmutation theory and rank for quantum braided groups*, 1993].

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Thank you for your attention!