

SPP Seminar:

Derived Symmetries and Monodromy

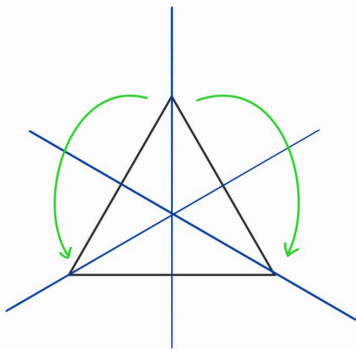
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Warm Up 1

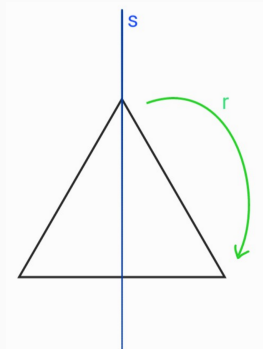
A **group** is an algebraic construction used to describe symmetries.

$$D_3 := \{s^i r^j \mid s^2 = 1, r^3 = 1, srs = r^{-1}\}$$



Let $(G, \star)$ be a group. An **action** of G on a space X is the association of a transformation $\rho(g) : X \rightarrow X$ to each $g \in G$ such that

$$\rho(g \star h) := \rho(g) \circ \rho(h).$$



A **(linear) representation** of G is a linear action of G :

We associate to each $g \in G$ a matrix $\rho(g) \in \text{Mat}_{n \times n}(\mathbb{C})$ such that

$$\rho(g \star h) = \rho(g) \cdot \rho(h).$$

Example

$$\rho(r) = \begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ -\sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Two representations are **isomorphic** if they are the same modulo a basis change.

	Basis = $\{(1, 0), (0, 1)\}$	Basis = $\{(1, 1), (-1, 1)\}$
$\rho(r)$	$\begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ -\sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}$	$\begin{pmatrix} -\sin(2\pi/3) & \cos(2\pi/3) \\ -\cos(2\pi/3) & -\sin(2\pi/3) \end{pmatrix}$
$\rho(s)$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

How can we show that two representations are not isomorphic?

Invariants

A matrix invariant is a property on matrices that do not change when we do a basis change.

The most common examples of matrix invariants are :

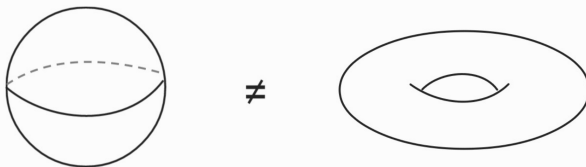
- the trace $g \mapsto \text{tr}(\rho(g))$ is called the **character** of ρ
- the determinant
- The eigenvalues
- The Jordan normal form.

Warm Up 2

Geometrical Invariants

A geometrical invariant is a property of geometrical objects (e.g. topological spaces, manifolds, algebraic varieties, etc...) that doesn't change modulo isomorphism.

Geometrical invariants are useful to show that two geometrical objects are different.



When considering an invariant, there are two important questions one should ask:

- How do we compute it?
- What geometrical property does the invariant "see"?

Example: the Euler Characteristic $\chi(S)$ of a surface S

How do we compute $\chi(S)$?

- Take a triangulation of S (i.e. embed a graph in S such that all faces are triangles)
- Compute $\# \text{faces} - \# \text{edges} + \# \text{vertices}$.

Examples:

Sphere : $\chi(S) = 2$

Torus : $\chi(S) = 0$

2-holes Torus : $\chi(S) = -2$

3-holes Torus : $\chi(S) = -4$

What does $\chi(S)$ sees?

It sees "how many holes" the surfaces has.

Cohomology, a powerful invariant

What does cohomology looks like?

X	$H^0(X)$	$H^1(X)$	$H^2(X)$	$H^3(X)$	$H^4(X)$	...
Sphere:	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	...
Torus :	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	0	0	...
2-holes Torus:	$\mathbb{Z}$	$\mathbb{Z}^4$	$\mathbb{Z}$	0	0	...
Projective Plane :	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	...
3-Sphere:	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	...

What does cohomology sees?

- Dimension of the space,
- "How many holes" it has,
- Orientability of the space,
- etc...

How do we compute cohomology?

It depends on the theory we're working on:

- Topology: We use simplexes (singular cohomology).
- Differential Geometry: We use differential forms (de Rham cohomology).
- Algebraic Geometry: We use sheaves (**sheaf cohomology**).

Let X be an algebraic variety:

- The derived category $D^b(X)$ is a mathematical object that "sees really well" the (sheaf) cohomology of X .
- Understanding $D^b(X)$ allows us hence to understand the invariants of X .
- In particular, **if we have symmetries of $D^b(X)$, this gives us symmetries on the invariants of X .**
→ It is interesting to study the symmetries of $D^b(X)$!

Conjecture (vague statement)

If X be a toric variety, then $D^b(X)$ has symmetries that can be described by a **specific differential equation**.

Questions:

- What do we mean by "symmetries described by a differential equation?"
- Which differential equations?

How does a differential equation describe symmetries?

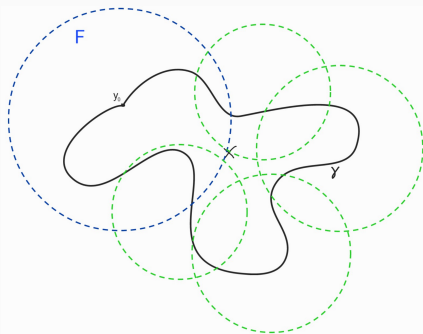
Let (E) be a complex linear differential equation over some space Y .

$$F^{(n)}(z) + p_1(z)F^{(n-1)}(z) + \dots + p_n(z)F(z) = 0$$

$(p_k(z) \text{ holomorphic on } Y).$

- The solutions are locally holomorphic.
- The set of solutions of (E) is a vector space.
- These solutions are local (and not necessarily defined on all of Y).

Let $y_0 \in Y$, $\gamma : [0, 1] \rightarrow Y$ be a loop satisfying $\gamma(0) = y_0 = \gamma(1)$, and F be a solution of (E) around y_0 .



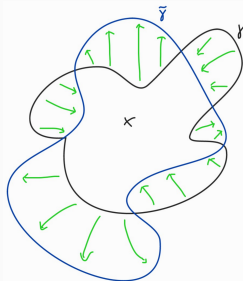
Then :

- We can do a analytic continuation of F along γ .
- When we come back to y_0 , we do not necessarily have **the same solution as before**.
- We call this new solution $h_\gamma(F)$.

Let V_{y_0} be the space of solutions of (E) around y_0 .

Then:

- $h_\gamma : V_{y_0} \rightarrow V_{y_0}$ is a linear map.
- For two loops γ, γ' , we have $h_\gamma \cdot h_{\gamma'} = h_{\gamma'}(F)(h_\gamma(F))$.
- If γ and $\tilde{\gamma}$ are homotopic, then $h_{\tilde{\gamma}}(F) = h_\gamma(F)$.



$\Rightarrow h$ is a representation of $\pi_1(Y)$.

We call this representation the **monodromy** of (E) .

Example 1

Let $Y := \mathbb{C} \setminus \{0\}$.

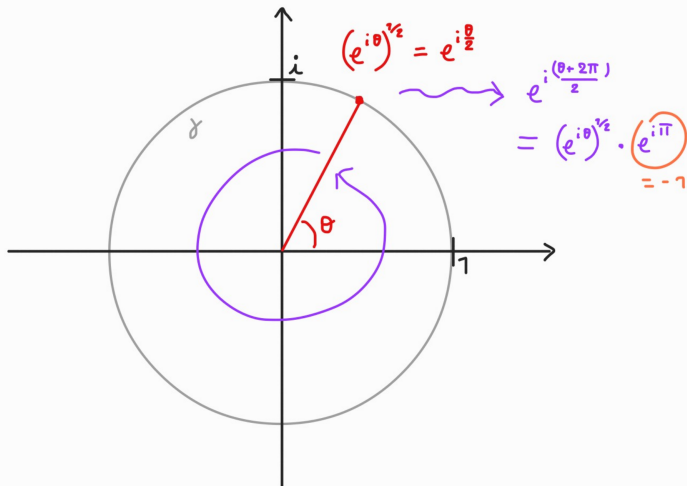
$\Rightarrow \pi_1(Y) \cong \mathbb{Z}$ is generated by

$$\gamma : [0, 1] \rightarrow Y : t \mapsto e^{2\pi it}.$$

Let (E) be the equation

$$F'(z) - \frac{1}{2z}F(z) = 0.$$

$$\Rightarrow \text{Sol} = \{z \mapsto C \cdot z^{1/2} \mid C \in \mathbb{C}\}$$



$$h_{\gamma}(C \cdot z^{1/2}) = -C \cdot z^{1/2}$$

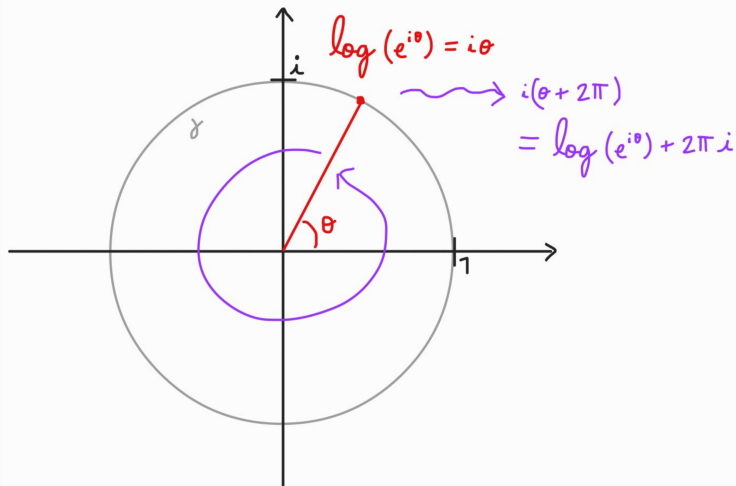
$$\Rightarrow h_{\gamma} = (-1)$$

Example 2

Let (E) be the equation

$$F'' + \frac{1}{z}F' = 0.$$

Then $Sol = \{C_1 \cdot \log(z) + C_2 \mid C_1, C_2 \in \mathbb{C}\}$.



If $e_1 = 2\pi i$, $e_2 = \log(z)$, then

$$h_\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Theorem (Fuchs)

Let (E) be a complex linear differential equation with regular singularities at $z_0 \in \mathbb{C}$:

Then the solutions of (E) are a combination of sums and products of

- Holomorphic functions around z_0 ,
- $(z - z_0)^\alpha$ ($\alpha \in \mathbb{R}$).
- $\log(z - z_0)^n$ ($n \in \mathbb{N}_0$).

only functions having singularities at $z = z_0$.

If moreover γ is a small loop around z_0 , and if the Jordan normal form of h_γ is

$$\begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix} \quad \text{where } J_k := \underbrace{\begin{pmatrix} a_k & 1 & & \\ & a_k & \ddots & \\ & & \ddots & 1 \\ & & & a_k \end{pmatrix}}_{n_k},$$

Then there are solutions of the form

holomorphic
around $z = z_0$

$$g(z) \cdot (z - z_0)^{\alpha_k} \cdot \log(z - z_0)^{n_k}.$$

$$e^{z \pi i \alpha_k} = a_k$$

(The normal Jordan form of) h_γ sees "how singular" the solution of (E) around z_0 are.

To recap

- Monodromy associates to each (complex linear) differential equation a linear representation.
- These are natural symmetries that appear on the space of solutions of (E) .
- Monodromy sees the what the singularities of the solutions look like.

Conjecture (more precise statement)

If X be a toric variety, then $D^b(X)$ has symmetries that **can be described by** a specific differential equation.

Conjecture (more precise statement)

If X be a toric variety, then $D^b(X)$ has symmetries that **are the monodromy of** a specific differential equation.

Which differential equation?

The equation that is conjectured to describe the symmetries of $D^b(X)$ is called a **GKZ system**.

It is a generalization of **Gauss Hypergeometric Equation**.

Definition:

Special functions are functions that cannot be defined using elementary functions (i.e. using polynomials, exponentials, trigonometric functions, etc...).

Examples

- Gauss error function: antiderivative of $\frac{2}{\sqrt{\pi}} e^{-x^2}$.
- Integral logarithm: antiderivative of $\frac{1}{\log(x)}$

Definition:

Hypergeometric functions are a family of special functions $F_{(a,b,c+1)}(z)$, $(a, b, c \in \mathbb{R})$ that appear as solutions of the differential equation:

$$z(D + a)(D + b)F = D(D + c)F, \quad (D := z \frac{d}{dz}).$$

Examples

- $\arcsin(z) = z \cdot F_{(\frac{1}{2}, \frac{1}{2}, \frac{3}{2})}(z)$.
- Legendre functions: $F_{(a, 1-a, c)}(z)$.
- etc...

→ Gauss hypergeometric equation

Definition:

The **Generalized Hypergeometric Equation** ${}_nF_{n-1}$ with parameters $(\alpha_1, \dots, \alpha_n; \beta_1 + 1, \dots, \beta_n + 1)$ is the equation

$$z(D + \alpha_1) \dots (D + \alpha_n) = (D + \beta_1) \dots (D + \beta_n)$$

"Definition":

A **GKZ system** is a PDE generalization of ${}_nF_{n-1}$.

Conjecture (precise statement)

If X be a toric variety, then $D^b(X)$ has symmetries that are the monodromy of a **specific differential equation**.

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If X be a toric variety, then $D^b(X)$ has symmetries that are the monodromy of a **GKZ system**.

Theorem

In a simpler case for X , $D^b(X)$ has symmetries that are the monodromy of a **Generalized Hypergeometric Equation**.

I was able to show it works in this simpler case.

Thank you for your attention ! :)