Isoperimetric inequalities for minimal surfaces of the hyperbolic space

Manh Tien NGUYEN

19/04/2022

Outline

Isoperimetric inequality

Minimal surfaces in hyperbolic space

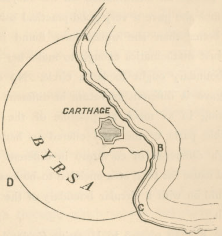
Knot theory

Queen Dido and the city of Carthage

From a lecture delivered by Lord Kelvin to the Royal Institution, 1893:

The city of Carthage

If the land is all of equal value the general solution of the problem shows that her line of ox-hide should be laid down in a circle. It shows also that if the sea is to be part of the boundary, starting, let us say, southward from any



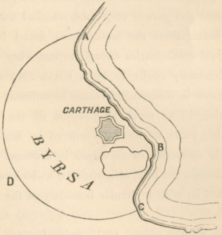
given point, A, of the coast, the inland bounding line must at its far end cut the coast line perpendicularly. Here, then, to complete our solution, we have a very curious and interesting, but not at all easy, geometrical question to

Queen Dido and the city of Carthage

From a lecture delivered by Lord Kelvin to the Royal Institution, 1893:

The city of Carthage

If the land is all of equal value the general solution of the problem shows that her line of ox-hide should be laid down in a circle. It shows also that if the sea is to be part of the boundary, starting, let us say, southward from any



given point, A, of the coast, the inland bounding line must at its far end cut the coast line perpendicularly. Here, then, to complete our solution, we have a very curious and interesting, but not at all easy, geometrical question to

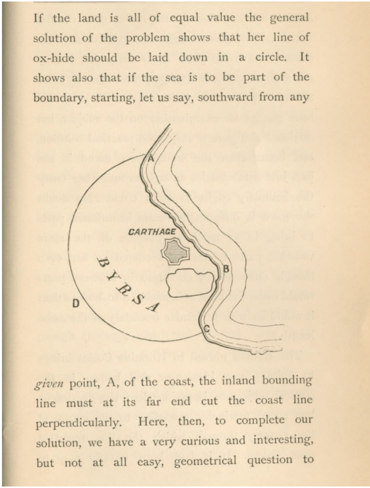
Queen Dido

- Fleed the Phoenician city of Tyre (Lebanon) to North Africa

Queen Dido and the city of Carthage

From a lecture delivered by Lord Kelvin to the Royal Institution, 1893:

The city of Carthage


If the land is all of equal value the general solution of the problem shows that her line of ox-hide should be laid down in a circle. It shows also that if the sea is to be part of the boundary, starting, let us say, southward from any

given point, A, of the coast, the inland bounding line must at its far end cut the coast line perpendicularly. Here, then, to complete our solution, we have a very curious and interesting, but not at all easy, geometrical question to

Queen Dido

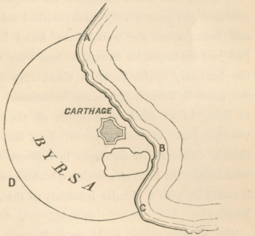
- Fleed the Phoenician city of Tyre (Lebanon) to North Africa
- grant from a native chief of as much land as she could enclose with an ox-hide

Queen Dido and the city of Carthage

From a lecture delivered by Lord Kelvin to the Royal Institution, 1893:

The city of Carthage

If the land is all of equal value the general solution of the problem shows that her line of ox-hide should be laid down in a circle. It shows also that if the sea is to be part of the boundary, starting, let us say, southward from any



given point, A, of the coast, the inland bounding line must at its far end cut the coast line perpendicularly. Here, then, to complete our solution, we have a very curious and interesting, but not at all easy, geometrical question to

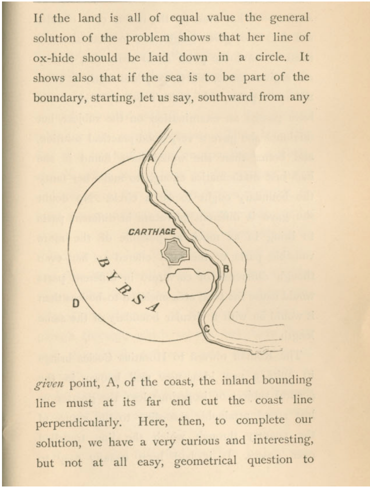
Queen Dido

- Fleed the Phoenician city of Tyre (Lebanon) to North Africa
- grant from a native chief of as much land as she could enclose with an ox-hide
- cut the hide into long strip, used the coastline, enclosed a circle

Queen Dido and the city of Carthage

From a lecture delivered by Lord Kelvin to the Royal Institution, 1893:

The city of Carthage


If the land is all of equal value the general solution of the problem shows that her line of ox-hide should be laid down in a circle. It shows also that if the sea is to be part of the boundary, starting, let us say, southward from any

given point, A, of the coast, the inland bounding line must at its far end cut the coast line perpendicularly. Here, then, to complete our solution, we have a very curious and interesting, but not at all easy, geometrical question to

Queen Dido

- Fleed the Phoenician city of Tyre (Lebanon) to North Africa
- grant from a native chief of as much land as she could enclose with an ox-hide
- cut the hide into long strip, used the coastline, enclosed a circle

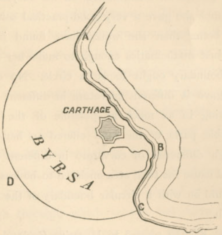
Question: Maximise the area enclosed by a given perimeter

Queen Dido and the city of Carthage

From a lecture delivered by Lord Kelvin to the Royal Institution, 1893:

The city of Carthage

If the land is all of equal value the general solution of the problem shows that her line of ox-hide should be laid down in a circle. It shows also that if the sea is to be part of the boundary, starting, let us say, southward from any



given point, A, of the coast, the inland bounding line must at its far end cut the coast line perpendicularly. Here, then, to complete our solution, we have a very curious and interesting, but not at all easy, geometrical question to

Queen Dido

- Fleed the Phoenician city of Tyre (Lebanon) to North Africa
- grant from a native chief of as much land as she could enclose with an ox-hide
- cut the hide into long strip, used the coastline, enclosed a circle

Question: Maximise the area enclosed by a given perimeter

$\longrightarrow$ Isoperimetric problem

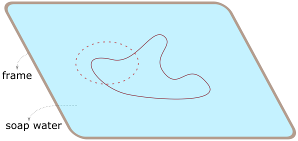
The mathematics of soap films

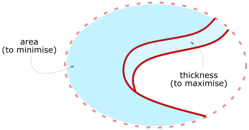
Video

The mathematics of soap films

Video

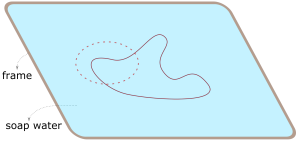
Experiment

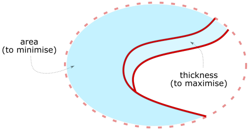




The mathematics of soap films

Video



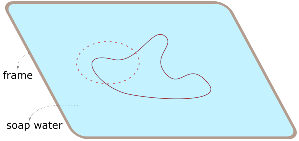


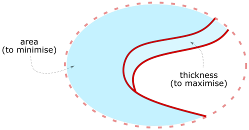
Experiment

- thickness of soap to maximise, area of soap to minimise

The mathematics of soap films

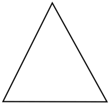
Video



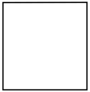


Experiment

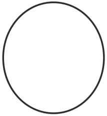
- thickness of soap to maximise, area of soap to minimise
- the hole solves isoperimetric problem



$A = \frac{\sqrt{3}}{36} = 0.0481$



$A = \frac{1}{16} = 0.0625$



$A = \frac{1}{4\pi} = 0.0796$

Isoperimetric inequality

Theorem

In the Euclidean plane, a curve of length L encloses an area A at most

$$A \leq \frac{L^2}{4\pi}.$$

Isoperimetric inequality

Theorem

In the Euclidean plane, a curve of length L encloses an area A at most

$$A \leq \frac{L^2}{4\pi}.$$

Equality happens only for round circles.

Isoperimetric inequality

Theorem

In the Euclidean plane, a curve of length L encloses an area A at most

$$A \leq \frac{L^2}{4\pi}.$$

Equality happens only for round circles.

In space-form of curvature $K = -1, 0, +1$,

$$4\pi A \leq L^2 + KA^2$$

Soap films minimise area

Isoperimetric inequality

Theorem

In the Euclidean plane, a curve of length L encloses an area A at most

$$A \leq \frac{L^2}{4\pi}.$$

Equality happens only for round circles.

In space-form of curvature $K = -1, 0, +1$,

$$4\pi A \leq L^2 + KA^2$$

Soap films minimise area

Minimal surfaces

= surfaces that locally minimise area

Isoperimetric inequality

Theorem

In the Euclidean plane, a curve of length L encloses an area A at most

$$A \leq \frac{L^2}{4\pi}.$$

Equality happens only for round circles.

In space-form of curvature $K = -1, 0, +1$,

$$4\pi A \leq L^2 + KA^2$$

Soap films minimise area

Minimal surfaces

= surfaces that locally minimise area

= vanishing mean curvature

Minimal surfaces

- research inititated by Lagrange (1762) who wrote down the equation and asked the Plateau problem.

Minimal surfaces

- research inititated by Lagrange (1762) who wrote down the equation and asked the Plateau problem.
- first examples (other than the plane) found by Meusnier (1776): the *catenoid* and helicoid

Minimal surfaces

- research inititated by Lagrange (1762) who wrote down the equation and asked the Plateau problem.
- first examples (other than the plane) found by Meusnier (1776): the *catenoid* and helicoid
- Weierstrass–Enneper (1863) generated minimal surfaces by holomorphic/meromorphic functions.

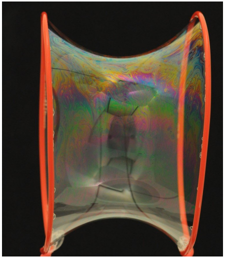
Minimal surfaces

- research inititated by Lagrange (1762) who wrote down the equation and asked the Plateau problem.
- first examples (other than the plane) found by Meusnier (1776): the *catenoid* and helicoid
- Weierstrass–Enneper (1863) generated minimal surfaces by holomorphic/meromorphic functions.
- In 1930, Douglas and Rado solved the Plateau problem.
- ...

Minimal surfaces

- research inititated by Lagrange (1762) who wrote down the equation and asked the Plateau problem.
- first examples (other than the plane) found by Meusnier (1776): the *catenoid* and helicoid
- Weierstrass–Enneper (1863) generated minimal surfaces by holomorphic/meromorphic functions.
- In 1930, Douglas and Rado solved the Plateau problem.
- ...

Catenoid



Catenary

Not catenoid

Properties of minimal surfaces

- Coordinates restrict to harmonic functions, i.e.

$$\Delta x_i = 0 \quad \text{on } \Sigma$$

Properties of minimal surfaces

- Coordinates restrict to harmonic functions, i.e.

$$\Delta x_i = 0 \quad \text{on } \Sigma$$

- Barrier (Maximum principle): Minimal surfaces cannot kiss a hyperplane $\longrightarrow$ No closed minimal surface.

Properties of minimal surfaces

- Coordinates restrict to harmonic functions, i.e.

$$\Delta x_i = 0 \quad \text{on } \Sigma$$

- Barrier (Maximum principle): Minimal surfaces cannot kiss a hyperplane $\longrightarrow$ No closed minimal surface.
- Monotonicity theorem

Isoperimetric inequality for minimal surfaces

Guess: Area and perimeter of a minimal surface in $\mathbb{R}^n$ satisfy $A \leq \frac{L^2}{4\pi}$.

Isoperimetric inequality for minimal surfaces

Guess: Area and perimeter of a minimal surface in $\mathbb{R}^n$ satisfy $A \leq \frac{L^2}{4\pi}$.

- Carleman (1921): minimal discs
- Reid (1959), Hsiung (1961): minimal surfaces with connected boundary
- Osserman–Schiffer (1975), Feinberg (1977): minimal annuli
- Li–Schoen–Yau (1984): *weakly connected* boundary
- Choe (1990): *radially connected* boundary
- Brendle (2020): codimension at most 2

Minimal surfaces in $\mathbb{H}^n$

The hyperbolic space $\mathbb{H}^n$

- simply-connected, constant sectional curvature -1

Minimal surfaces in $\mathbb{H}^n$

The hyperbolic space $\mathbb{H}^n$

- simply-connected, constant sectional curvature -1
- has totally geodesic copies of $\mathbb{H}^{n-1}$

Minimal surfaces in $\mathbb{H}^n$

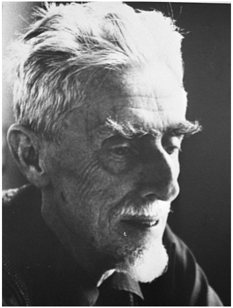
The hyperbolic space $\mathbb{H}^n$

- simply-connected, constant sectional curvature -1
- has totally geodesic copies of $\mathbb{H}^{n-1}$

Escher's Heaven and Hell

(Circle Limit IV)

Figure: M. C. Escher



Poincaré ball model

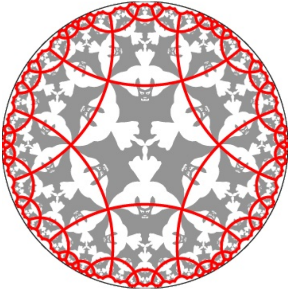
In $\mathbb{B}^n$, r: Euclidean distance to centre,

$$g_H = \frac{4}{(1-r^2)^2} g_{\mathrm{Euclidean}}$$

Poincaré ball model

In $\mathbb{B}^n$, r: Euclidean distance to centre,

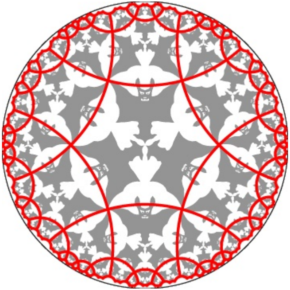
$$g_H = \frac{4}{(1-r^2)^2} g_{\mathrm{Euclidean}}$$



Poincaré ball model

In $\mathbb{B}^n$, r: Euclidean distance to centre,

$$g_H = \frac{4}{(1-r^2)^2} g_{\mathrm{Euclidean}}$$



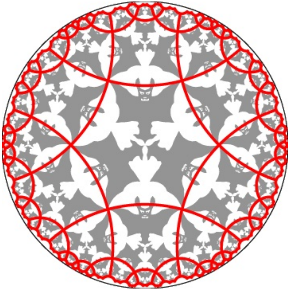
Minimal surfaces

- Barrier: Minimal surfaces cannot kiss copies of $\mathbb{H}^{n-1}$ hyperplane
  $\longrightarrow$ No closed minimal surfaces in $\mathbb{H}^n$

Poincaré ball model

In $\mathbb{B}^n$, r: Euclidean distance to centre,

$$g_H = \frac{4}{(1-r^2)^2} g_{\mathrm{Euclidean}}$$



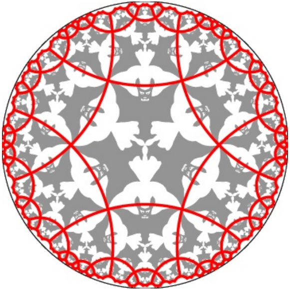
Minimal surfaces

- Barrier: Minimal surfaces cannot kiss copies of $\mathbb{H}^{n-1}$ hyperplane
  $\longrightarrow$ No closed minimal surfaces in $\mathbb{H}^n$
- Asymptotic Plateau problem solved by M. Anderson (1982):

Poincaré ball model

In $\mathbb{B}^n$, r: Euclidean distance to centre,

$$g_H = \frac{4}{(1-r^2)^2} g_{\mathrm{Euclidean}}$$



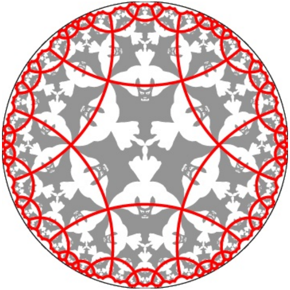
Minimal surfaces

- Barrier: Minimal surfaces cannot kiss copies of $\mathbb{H}^{n-1}$ hyperplane $\longrightarrow$ No closed minimal surfaces in $\mathbb{H}^n$
- Asymptotic Plateau problem solved by M. Anderson (1982):Given an immersed curve γ of S^{n-1},

Poincaré ball model

In $\mathbb{B}^n$, r: Euclidean distance to centre,

$$g_H = \frac{4}{(1-r^2)^2} g_{\text{Euclidean}}$$



Minimal surfaces

- Barrier: Minimal surfaces cannot kiss copies of $\mathbb{H}^{n-1}$ hyperplane $\longrightarrow$ No closed minimal surfaces in $\mathbb{H}^n$
- Asymptotic Plateau problem solved by M. Anderson (1982):Given an immersed curve γ of S^{n-1},there is an *area-minimising minimal surface* of $\mathbb{H}^n$ that is asymptotic to γ.

The half-space model

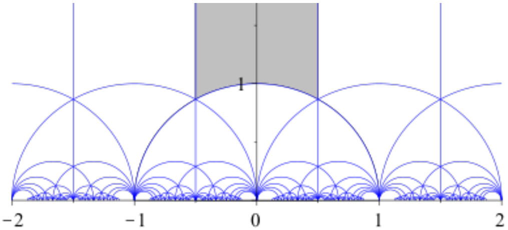
$\mathbb{R}_{>0(x)} \times \mathbb{R}^{n-1}_{(y_1,\ldots y_{n-1})}$ with the metric

$$g_H = \frac{g_{\text{Euclidean}}}{x^2}$$

The half-space model

$\mathbb{R}_{>0(x)} \times \mathbb{R}^{n-1}_{(y_1,\ldots y_{n-1})}$ with the metric

$$g_H = \frac{g_{\mathrm{Euclidean}}}{x^2}$$



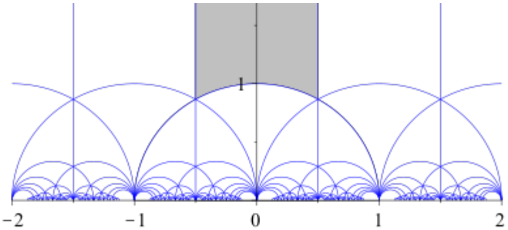
Facts

- half spheres and vertical planes are copies of $\mathbb{H}^{n-1}$

The half-space model

$\mathbb{R}_{>0(x)} \times \mathbb{R}^{n-1}_{(y_1,\dots y_{n-1})}$ with the metric

$$g_H = \frac{g_{\mathrm{Euclidean}}}{x^2}$$



Facts

- half spheres and vertical planes are copies of $\mathbb{H}^{n-1}$
- horizontal planes are horocycles/horospheres

Hyperboloid model and Minkowskian coordinates

- Minkowskian space $\mathbb{R}^{n,1}$ with metric $g = d\xi_1^2 + \ldots d\xi_n^2 - d\xi_0^2$.

Hyperboloid model and Minkowskian coordinates

- Minkowskian space $\mathbb{R}^{n,1}$ with metric $g = d\xi_1^2 + \ldots d\xi_n^2 - d\xi_0^2$.
- Unit hyperboloid $H: \ \xi_0^2 = 1 + \xi_1^2 + \cdots + \xi_n^2$

Hyperboloid model and Minkowskian coordinates

- Minkowskian space $\mathbb{R}^{n,1}$ with metric $g = d\xi_1^2 + \dots d\xi_n^2 - d\xi_0^2$.
- Unit hyperboloid $H: \ \xi_0^2 = 1 + \xi_1^2 + \cdots + \xi_n^2$

Coordinates	Geometric object	Function	Level sets
Time	interior point	$\xi_0 = \cosh d$	circles

Hyperboloid model and Minkowskian coordinates

- Minkowskian space $\mathbb{R}^{n,1}$ with metric $g = d\xi_1^2 + \dots d\xi_n^2 - d\xi_0^2$.
- Unit hyperboloid $H: \ \xi_0^2 = 1 + \xi_1^2 + \cdots + \xi_n^2$

Coordinates	Geometric object	Function	Level sets
Time	interior point	$\xi_0 = \cosh d$	circles
Space	copy of $\mathbb{H}^{n-1}$	$\xi_1 = \sinh d$	hypercycles

Hyperboloid model and Minkowskian coordinates

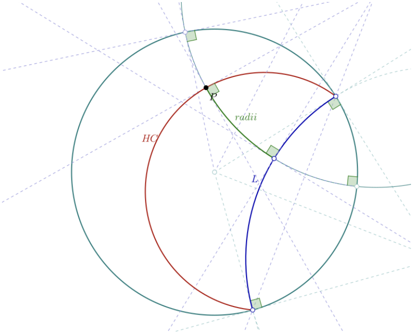
- Minkowskian space $\mathbb{R}^{n,1}$ with metric $g = d\xi_1^2 + \ldots d\xi_n^2 - d\xi_0^2$.
- Unit hyperboloid $H: \ \xi_0^2 = 1 + \xi_1^2 + \cdots + \xi_n^2$

Coordinates	Geometric object	Function	Level sets
Time	interior point	$\xi_0 = \cosh d$	circles
Space	copy of $\mathbb{H}^{n-1}$	$\xi_1 = \sinh d$	hypercycles
Null	boundary point	$\xi_l = \frac{1}{x} (= \exp \beta)$	horocycles

Hyperboloid model and Minkowskian coordinates

- Minkowskian space $\mathbb{R}^{n,1}$ with metric $g = d\xi_1^2 + \dots d\xi_n^2 - d\xi_0^2$.
- Unit hyperboloid $H: \ \xi_0^2 = 1 + \xi_1^2 + \cdots + \xi_n^2$

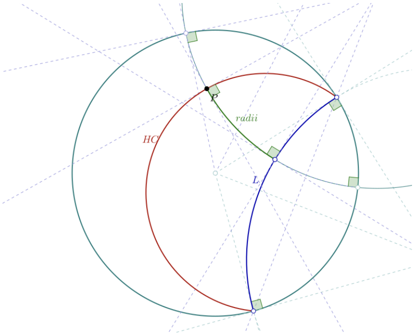
Coordinates	Geometric object	Function	Level sets
Time	interior point	$\xi_0 = \cosh d$	circles
Space	copy of $\mathbb{H}^{n-1}$	$\xi_1 = \sinh d$	hypercycles
Null	boundary point	$\xi_l = \frac{1}{x} (= \exp \beta)$	horocycles



Hyperboloid model and Minkowskian coordinates

- Minkowskian space $\mathbb{R}^{n,1}$ with metric $g = d\xi_1^2 + \ldots d\xi_n^2 - d\xi_0^2$.
- Unit hyperboloid $H: \ \xi_0^2 = 1 + \xi_1^2 + \cdots + \xi_n^2$

Coordinates	Geometric object	Function	Level sets
Time	interior point	$\xi_0 = \cosh d$	circles
Space	copy of $\mathbb{H}^{n-1}$	$\xi_1 = \sinh d$	hypercycles
Null	boundary point	$\xi_l = \frac{1}{x} (= \exp \beta)$	horocycles



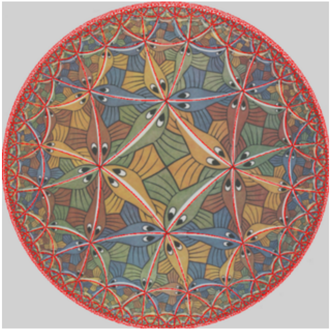


Figure: Escher's Fish (Circle Limit III)

Graham–Witten's renormalised area

- Any complete minimal surface Σ of $\mathbb{H}^n$ has infinite area (recall: no closed minimal surface).

Graham–Witten's renormalised area

- Any complete minimal surface Σ of $\mathbb{H}^n$ has infinite area (recall: no closed minimal surface).Goal: get a finite number out of it.

Graham–Witten's renormalised area

- Any complete minimal surface Σ of $\mathbb{H}^n$ has infinite area (recall: no closed minimal surface).Goal: get a finite number out of it.
- Fix a half space coordinate x. Let

$$A_\epsilon = \mathrm{Area}(\Sigma \cap \{x \geq \epsilon\})$$

Graham–Witten's renormalised area

- Any complete minimal surface Σ of $\mathbb{H}^n$ has infinite area (recall: no closed minimal surface). Goal: get a finite number out of it.
- Fix a half space coordinate x. Let

$$A_\epsilon = \mathrm{Area}(\Sigma \cap \{x \geq \epsilon\})$$

Theorem (Graham–Witten '99)

If Σ is minimal then

$$A_\epsilon = \frac{L}{\epsilon} + \mathcal{A}_R + O(\epsilon)$$

where L is the Euclidean length of $\partial\Sigma$.

Graham–Witten's renormalised area

- Any complete minimal surface Σ of $\mathbb{H}^n$ has infinite area (recall: no closed minimal surface).Goal: get a finite number out of it.
- Fix a half space coordinate x. Let

$$A_\epsilon = \mathrm{Area}(\Sigma \cap \{x \geq \epsilon\})$$

Theorem (Graham–Witten '99)

If Σ is minimal then

$$A_\epsilon = \frac{L}{\epsilon} + \mathcal{A}_R + O(\epsilon)$$

where L is the Euclidean length of $\partial\Sigma$.
Moreover, $\mathcal{A}_R$ is independent of the choice of the boundary defining function x.

Isoperimetric inequality

Theorem (Bernstein, T.)

Let Σ be a minimal surface of $\mathbb{H}^n$ bounded by a curve $\gamma \subset S^{n-1}$.

Isoperimetric inequality

Theorem (Bernstein, T.)

Let Σ *be a minimal surface of* $\mathbb{H}^n$ *bounded by a curve* $\gamma \subset S^{n-1}$*. Then*

$$\mathcal{A}_R(\Sigma) + \sup_{round\ g} |\gamma|_g \leq 0.$$

Equality happens only for totally geodesic copies of $\mathbb{H}^2$*.*

Isoperimetric inequality

Theorem (Bernstein, T.)

Let Σ be a minimal surface of $\mathbb{H}^n$ bounded by a curve $\gamma \subset S^{n-1}$. Then

$$\mathcal{A}_R(\Sigma) + \sup_{round\ g} |\gamma|_g \leq 0.$$

Equality happens only for totally geodesic copies of $\mathbb{H}^2$.

Remark:

interior points $\longleftrightarrow$ time coordinates $\longleftrightarrow$ round metrics

Isoperimetric inequality

Theorem (Bernstein, T.)

Let Σ be a minimal surface of $\mathbb{H}^n$ bounded by a curve $\gamma \subset S^{n-1}$. Then

$$\mathcal{A}_R(\Sigma) + \sup_{round\ g} |\gamma|_g \leq 0.$$

Equality happens only for totally geodesic copies of $\mathbb{H}^2$.

Remark:

interior points $\longleftrightarrow$ time coordinates $\longleftrightarrow$ round metrics
boundary points $\longleftrightarrow$ null coordinates

Isoperimetric inequality

Theorem (Bernstein, T.)

Let Σ be a minimal surface of $\mathbb{H}^n$ bounded by a curve $\gamma \subset S^{n-1}$. Then

$$\mathcal{A}_R(\Sigma) + \sup_{round\ g} |\gamma|_g \leq 0.$$

Equality happens only for totally geodesic copies of $\mathbb{H}^2$.

Remark:

interior points $\longleftrightarrow$ time coordinates $\longleftrightarrow$ round metrics
boundary points $\longleftrightarrow$ null coordinates $\longleftrightarrow$ flat metrics

Isoperimetric inequality

Theorem (Bernstein, T.)

Let Σ be a minimal surface of $\mathbb{H}^n$ bounded by a curve $\gamma \subset S^{n-1}$. Then

$$\mathcal{A}_R(\Sigma) + \sup_{round\ g} |\gamma|_g \leq 0.$$

Equality happens only for totally geodesic copies of $\mathbb{H}^2$.

Remark:

interior points $\longleftrightarrow$ time coordinates $\longleftrightarrow$ round metrics
boundary points $\longleftrightarrow$ null coordinates $\longleftrightarrow$ flat metrics
copies of $\mathbb{H}^{n-1}$ $\longleftrightarrow$ space coordinates

Isoperimetric inequality

Theorem (Bernstein, T.)

Let Σ be a minimal surface of $\mathbb{H}^n$ bounded by a curve $\gamma \subset S^{n-1}$. Then

$$\mathcal{A}_R(\Sigma) + \sup_{round\ g} |\gamma|_g \leq 0.$$

Equality happens only for totally geodesic copies of $\mathbb{H}^2$.

Remark:

interior points $\longleftrightarrow$ time coordinates $\longleftrightarrow$ round metrics
boundary points $\longleftrightarrow$ null coordinates $\longleftrightarrow$ flat metrics
copies of $\mathbb{H}^{n-1}$ $\longleftrightarrow$ space coordinates $\longleftrightarrow$ doubled hyperbolic metrics

Isoperimetric inequalities

Theorem (T.)

1. *If* $\xi_0 \geq a \geq 1$ *on* Σ *then*

Isoperimetric inequalities

Theorem (T.)

1. *If* $\xi_0 \geq a \geq 1$ *on* Σ *then*

$$\mathcal{A}_R(\Sigma) + \frac{1}{2}|\gamma|_{g_o}\left(a + \frac{1}{a}\right) \leq 0 \qquad (1)$$

Isoperimetric inequalities

Theorem (T.)

1. *If* $\xi_0 \geq a \geq 1$ *on* Σ *then*

$$\mathcal{A}_R(\Sigma) + \frac{1}{2}|\gamma|_{g_0}\left(a + \frac{1}{a}\right) \leq 0 \tag{1}$$

2. *If* $\xi_1 \geq a > 0$ *on* Σ *then*

$$\mathcal{A}_R(\Sigma) + \frac{1}{2}|\gamma|_{g_1}\left(a - \frac{1}{a}\right) \leq 0 \tag{2}$$

3. *If* $\xi_l \geq a > 0$ *on* Σ *then*

$$\mathcal{A}_R(\Sigma) + \frac{1}{2}|\gamma|_{g_l} a \leq 0 \tag{3}$$

Here ξ_0, ξ_1, ξ_l be Minkowskian coordinates.

Under the hood: Monotonicity theorems

Under the hood: Monotonicity theorems

Theorem (T.)

Each Minkowskian coordinate ξ *gives a monotonicity theorem.*

Under the hood: Monotonicity theorems

Theorem (T.)

Each Minkowskian coordinate ξ gives a monotonicity theorem.

- Those are statements about the area of the minimal surface between level sets of ξ.

Under the hood: Monotonicity theorems

Theorem (T.)

Each Minkowskian coordinate ξ gives a monotonicity theorem.

- Those are statements about the area of the minimal surface between level sets of ξ.
- More general: warped spaces, manifolds with curvature bounded from above.

Counting minimal surfaces of $\mathbb{H}^n$ bounded by a knot/link

- Tomi–Tromba ('78): solution of the embedded Plateau problem

Counting minimal surfaces of $\mathbb{H}^n$ bounded by a knot/link

- Tomi–Tromba ('78): solution of the embedded Plateau problem
- White ('87) counting minimal surfaces of $\mathbb{R}^n$, modulo properness

Counting minimal surfaces of $\mathbb{H}^n$ bounded by a knot/link

- Tomi–Tromba ('78): solution of the embedded Plateau problem
- White ('87) counting minimal surfaces of $\mathbb{R}^n$, modulo properness
- Alexakis–Mazzeo ('10): counting embedded minimal surfaces of $\mathbb{H}^3$

Counting minimal surfaces of $\mathbb{H}^n$ bounded by a knot/link

- Tomi–Tromba ('78): solution of the embedded Plateau problem
- White ('87) counting minimal surfaces of $\mathbb{R}^n$, modulo properness
- Alexakis–Mazzeo ('10): counting embedded minimal surfaces of $\mathbb{H}^3$
- Fine ('21): counting *branched minimally immersed surfaces* of $\mathbb{H}^4$

Counting minimal surfaces of $\mathbb{H}^n$ bounded by a knot/link

- Tomi–Tromba ('78): solution of the embedded Plateau problem
- White ('87) counting minimal surfaces of $\mathbb{R}^n$, modulo properness
- Alexakis–Mazzeo ('10): counting embedded minimal surfaces of $\mathbb{H}^3$
- Fine ('21): counting *branched minimally immersed surfaces* of $\mathbb{H}^4$

Theorem (T.)

Let $L = L_1 \sqcup L_2$ be a separated union of 2 links of S^3.

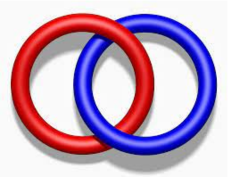
Counting minimal surfaces of $\mathbb{H}^n$ bounded by a knot/link

- Tomi–Tromba ('78): solution of the embedded Plateau problem
- White ('87) counting minimal surfaces of $\mathbb{R}^n$, modulo properness
- Alexakis–Mazzeo ('10): counting embedded minimal surfaces of $\mathbb{H}^3$
- Fine ('21): counting *branched minimally immersed surfaces* of $\mathbb{H}^4$

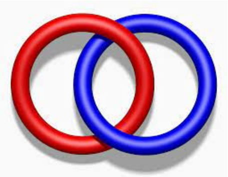
Theorem (T.)

Let $L = L_1 \sqcup L_2$ be a separated union of 2 links of S^3.Can rearrange L so that there is no connected minimal surfaces of $\mathbb{H}^4$ filling it.

Surfaces filling Hopf links



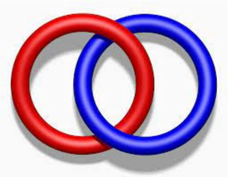
Surfaces filling Hopf links



Theorem (T.)

There are minimal annuli of $\mathbb{H}^4$ *filling the Hopf links of* S^3.

Surfaces filling Hopf links

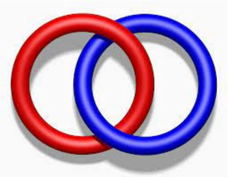


Theorem (T.)

There are minimal annuli of $\mathbb{H}^4$ *filling the Hopf links of* S^3.

These annuli are surfaces of revolution.

Surfaces filling Hopf links

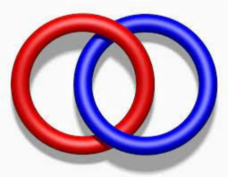


Theorem (T.)

There are minimal annuli of $\mathbb{H}^4$ filling the Hopf links of S^3.

These annuli are surfaces of revolution. The rotation here is by changing the 2 complex coordinates of $\mathbb{H}^4 \cong \mathbb{B}^4 \subset \mathbb{C}^2$ by an opposite phase.

Surfaces filling Hopf links



Theorem (T.)

There are minimal annuli of $\mathbb{H}^4$ filling the Hopf links of S^3.

These annuli are surfaces of revolution. The rotation here is by changing the 2 complex coordinates of $\mathbb{H}^4 \cong \mathbb{B}^4 \subset \mathbb{C}^2$ by an opposite phase.

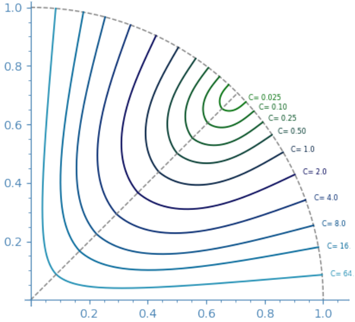


Figure: the new "catenary"