

Characterization of Hopf Categories in $\text{Vect}_{\mathbb{K}}$

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- ▶ $\mathbb{K}[X]$ denotes the ring of polynomials.
- ▶ $\delta_{ab} = 0, a \neq b$ and $\delta_{ab} = 1, a = b$.

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Definitions

Definition

A group G is a set with maps $m : G \times G \rightarrow G$, $1_G : \{*\} \rightarrow G$ and $S : G \rightarrow G$ with m being associative and satisfying

$$m \circ (1_G \times Id_G) = m \circ (Id_G \times 1_G) = Id_G$$

$$m \circ (Id_G \times S) \circ \Delta(g) = m \circ (S \times Id_G) \circ \Delta(g) = 1_G \circ \varepsilon(g)$$

where $\varepsilon(g) : G \rightarrow \{*\}$ and $\Delta : G \rightarrow G \times G$ is given by $\Delta(g) = g \times g$.

Obs:

$$(\Delta \times Id_G) \circ \Delta = (Id \times \Delta) \circ \Delta,$$

$$Id_G = (\varepsilon \times Id_G) \circ \Delta = (Id_G \times \varepsilon) \circ \Delta$$

$$(m \times m) \circ (Id_G \times \tau \times Id_G) \circ (\Delta \times \Delta) = \Delta \circ m$$

where $\tau : G \times G \rightarrow G \times G$ is given by $\tau(g \times h) = h \times g$.

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A connected groupoid is given by a set of indexes X , a collection of sets $G_{x,y}$ ($x, y \in X$), a family of maps $m_{x,y,z} : G_{x,y} \times G_{y,z}$ (multiplication/composition maps), map $1_x : \{*\} \rightarrow G_{x,x}$ and $s_{x,y} : G_{x,y} \rightarrow G_{y,x}$, $\varepsilon_{x,y} : G_{x,y} \rightarrow \{*\}$ that satisfy the following

$$m_{x,x,y} \circ (1_{x,x} \times Id_{G_{x,y}}) = m_{x,y,y} \circ (Id_{G_{x,y}} \times 1_{y,y}) = Id_{G_{x,y}}$$

$$m_{y,x,y} \circ (s_{x,y} \times Id_{G_{x,y}}) \circ \Delta_{x,y} = 1_y \circ \varepsilon_{x,y},$$

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where $\Delta_{x,y} : G_{x,y} \rightarrow G_{x,y} \times G_{x,y}$ is given by $\Delta_{x,y}(g) = g \otimes g$ for $g \in G_{x,y}$.

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Example: V_1, V_2, \dots, V_m \mathbb{R} -vector spaces,
 $\dim_{\mathbb{R}} V_1 = \dim_{\mathbb{R}} V_2 = \dots = \dim_{\mathbb{R}} V_m = n \in \mathbb{N}$ Then we have a connected
groupoid given by:

- ▶ The set X is given by $X = \{V_1, V_2, \dots, V_m\}$.
- ▶ We have $\mathcal{H}_{V_i, V_j} = \{F : V_i \rightarrow V_j; F \text{ is a bijective } \mathbb{R}\text{-linear map}\},$

$$m_{V_i V_j V_k}(F \times G) = G \circ F$$

$$1_{V_i} = Id_{V_i}$$

$$S_{V_i V_j}(F) = F^{-1}$$

Definition of Algebra

Definition

An algebra $(A, m, 1)$ is a \mathbb{K} -vector space A with a multiplication map $m : A \otimes A \rightarrow A$ (Notation: $m(a \otimes b) = ab$, $a, b \in A$), and $1 : \mathbb{K} \rightarrow A$ such that

$$(ab)c = a(bc),$$

$$1a = a1 = a,$$

for $a, b, c \in A$.

Example

- ▶ $A = \mathbb{K}[X]$ (the algebra of polynomials).
- ▶ $A = \mathbb{K}G$ where M is a group with $m(g \otimes h) = gh$ and $1 = e$.
- ▶ If A and B are \mathbb{K} -algebras, then $A \otimes B$ is a \mathbb{K} -algebra with $1_{A \otimes B} = 1_A \otimes 1_B$, and $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.
- ▶ If A is a \mathbb{K} -algebra, then A^{op} is a \mathbb{K} -algebra, where $A = A^{op}$ as \mathbb{K} -v.s., $1_{A^{op}} = 1_A$ and $a \cdot b = ba$ for $a, b \in A$.

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Definition

A coalgebra (C, Δ, ε) is a \mathbb{K} -vector space C with a comultiplication map $\Delta : C \rightarrow C \otimes C$ (notation: $\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$) and counit map $\varepsilon : C \rightarrow \mathbb{K}$ such that:

$$\sum_{(c)} \sum_{(c_{(1)})} c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = \sum_{(c)} \sum_{(c_{(2)})} c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)},$$

and

$$\sum_{(c)} \varepsilon(c_{(1)})c_{(2)} = \sum_{(c)} c_{(1)}\varepsilon(c_{(2)}) = c$$

for all $c \in C$.

Example

Let G be a group. Then, we have a coalgebra which is given by

▶ $C = \mathbb{K}G.$



$$\Delta(g) = g \otimes g,$$

$g \in G.$



$$\varepsilon(g) = 1, g \in G.$$

Example

▶ If C, D are \mathbb{K} -coalgebras, then $C \otimes D$ is a \mathbb{K} -coalgebra, where

$$\Delta(c \otimes d) = \sum_{(c),(d)} c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)} \text{ and}$$
$$\varepsilon_{C \otimes D}(c \otimes d) = \varepsilon(c)\varepsilon(d), \text{ for } c \in C, d \in D.$$

▶ If C is a coalgebra, then C^{cop} is a \mathbb{K} -coalgebra, where $C^{\text{cop}} = C$ as a \mathbb{K} -vector space and $\varepsilon_{C^{\text{cop}}} = \varepsilon_C$, with $\Delta_{C^{\text{cop}}}(c) = \sum_{(c)} c_{(2)} \otimes c_{(1)}$ for $c \in C$.

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Definition

A Hopf algebra $(H, m, 1, \Delta, \varepsilon, S)$ is given by a \mathbb{K} -vector space H such that $(H, m, 1)$ is a \mathbb{K} -algebra, (H, Δ, ε) is a \mathbb{K} -coalgebra such that

$$\sum_{(ab)} (ab)_{(1)} \otimes (ab)_{(2)} = \sum_{(a)} \sum_{(b)} a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)},$$

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b),$$

$$\sum_{(1)} 1_{(1)} \otimes 1_{(2)} = 1 \otimes 1,$$

$$\varepsilon(1) = 1$$

$$\sum_{(a)} S(a_{(1)})a_{(2)} = \sum_{(a)} a_{(1)}S(a_{(2)}) = 1_H\varepsilon(a),$$

for $a, b \in H$.

Here, we always assume S is invertible.

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- $\mathbb{K}G$ where G is a group.

$$m(g \otimes h) = gh$$

$$\Delta(g) = g \otimes g$$

$$\varepsilon(g) = 1$$

$$S(g) = g^{-1}.$$

- $\mathbb{K}G^* = \text{Hom}(\mathbb{K}G, \mathbb{K})$ for G a finite group.

$$m(g^* \otimes h^*) = \delta_{gh} g^* = \delta_{gh} h^*$$

$$\Delta(g^*) = \sum_{h \in G} h^* \otimes (h^{-1}g)^*,$$

$$1_{\mathbb{K}G^*} = \sum_{g \in G} g^*$$

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for $g, h \in G$,

$$H = \mathbb{K}[X],$$

$$m(X^i \otimes X^j) = X^{i+j}$$

$$\Delta(X^i) = (X \otimes 1 + 1 \otimes X)^i$$

$$S(1) = 1$$

$$S(X^i) = (-1)^i X^i$$

$$\varepsilon(1) = 1, \varepsilon(X^i) = 0$$

$$i \in \mathbb{N}.$$

Take $\alpha \in \{0, \pi/2, \pi, 3\pi/2\}$,

$$H_\alpha = \langle 1, s, c, s^2 \rangle$$

commutative with relations

$$c^2 + s^2 = 1, cs = 0,$$

with

$$\Delta(c) = \cos(\alpha)(c \otimes c - s \otimes s) - \sin(\alpha)(c \otimes s + s \otimes c)$$

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Definition

A Hopf category \mathcal{H} is given by a set of indexes X , a collection of \mathbb{K} -coalgebras $(\mathcal{H}_{x,y}, \Delta_{x,y}, \varepsilon_{x,y})$ and a family of maps $m_{x,y,z} : \mathcal{H}_{x,y} \otimes \mathcal{H}_{y,z} \rightarrow \mathcal{H}_{x,z}$, $1_x : \mathbb{K} \rightarrow \mathcal{H}_{x,x}$ and (notation: $m_{x,y,z}(c^{x,y} \otimes d^{y,z}) = c^{x,y} d^{y,z}$ $s_{x,y} : \mathcal{H}_{x,y} \rightarrow \mathcal{H}_{y,x}$ satisfying

$$(c^{x,y} d^{y,z}) f^{z,w} = c^{x,y} (d^{y,z} f^{z,w}),$$

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$$\sum_{(c^{x,y} d^{y,z})} (c^{x,y} d^{y,z})_{(1)} \otimes (c^{x,y} d^{y,z})_{(2)} = \sum_{(c^{x,y})} \sum_{(d^{y,z})} c_{(1)}^{x,y} d_{(1)}^{y,z} \otimes c_{(2)}^{x,y} d_{(2)}^{y,z},$$

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$$(c^{x,y} d^{y,z}) f^{z,w} = c^{x,y} (d^{y,z} f^{z,w}),$$

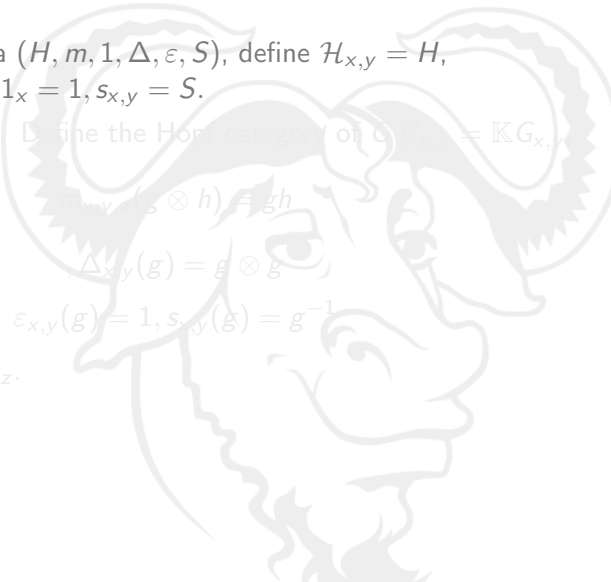
$$c^{x,y} 1_{y,y} = 1_{x,x} c^{x,y} = c^{x,y},$$

$$\sum_{(c^{x,y} d^{y,z})} (c^{x,y} d^{y,z})_{(1)} \otimes (c^{x,y} d^{y,z})_{(2)} = \sum_{(c^{x,y})} \sum_{(d^{y,z})} c_{(1)}^{x,y} d_{(1)}^{y,z} \otimes c_{(2)}^{x,y} d_{(2)}^{y,z},$$

$$\sum_{(1_x)} (1_x)_{(1)} \otimes \sum_{(1_x)} (1_x)_{(2)} = 1_x \otimes 1_x,$$

$$\varepsilon_{x,z}(c^{x,y} d^{y,z}) = \varepsilon_{x,y}(c^{x,y}) \varepsilon_{y,z}(d^{y,z}),$$

$$\sum s_{x,y}(c_{(1)}^{x,y}) c_{(2)}^{x,y} = \varepsilon_{x,y}(c^{x,y}) 1_y, \quad \sum c_{(1)}^{x,y} s_{x,y}(c_{(2)}^{x,y}) = \varepsilon(c^{x,y}) 1_x$$

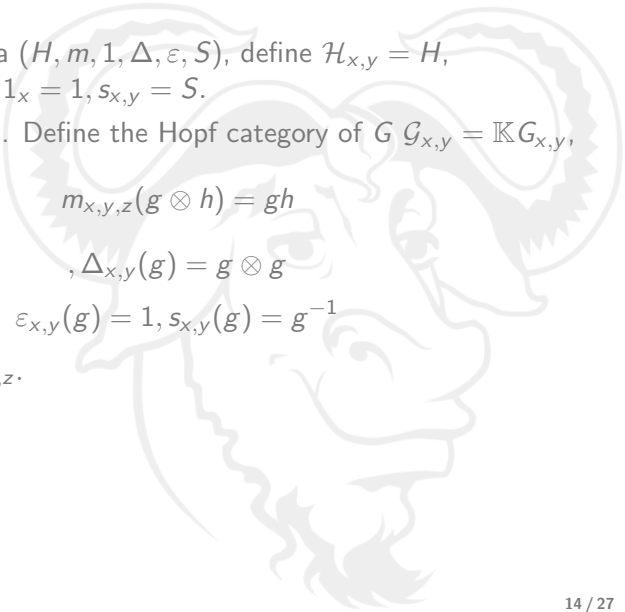
- 
- ▶ Given a Hopf algebra $(H, m, 1, \Delta, \varepsilon, S)$, define $\mathcal{H}_{x,y} = H$,
 $m = m_H, \Delta_{x,y} = \Delta, 1_x = 1, s_{x,y} = S$.
 - ▶ Let G be a groupoid. Define the Hopf category of G $\mathcal{G}_{x,y} = \mathbb{K}G_x$,

$$m_{x,y}(g \otimes h) = gh$$

$$\Delta_{x,y}(g) = g \otimes g$$

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for $g \in G_{x,y}, h \in G_{y,z}$.

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Let G be a connected groupoid. Then we know:

- ▶ $G_{x,x} \cong G_{y,y}$ as groups.
- ▶ If we know $G_{x,y}$, we know $G_{y,x}$.
- ▶ If we know $G_{x,y}$ and $G_{y,z}$, then we know $G_{x,z}$.

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Problem: Characterizing Hopf categories.

- ▶ For a Hopf category \mathcal{H} , given $\mathcal{H}_{x,x}$ and $\mathcal{H}_{x,y}$, what can be said about $\mathcal{H}_{y,y}$ for $x, y \in X$?
- ▶ Knowing $\mathcal{H}_{x,y}$ and $\mathcal{H}_{y,z}$, what can be said about $\mathcal{H}_{x,z}$?
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Definition

Let H be a Hopf algebra over \mathbb{K} . We say that A is a left H -module if there is a map $\cdot : H \otimes A \rightarrow A$ such that

$$\cdot \circ (m_H \otimes A) = \cdot \circ (H \otimes \cdot)$$

Notation: $\cdot(a \otimes h) = a \cdot h$.

Definition

A left H module A is a left H -Hopf-Galois co-object if

- ▶ (A, Δ, ε) is a coalgebra.
- ▶ $h_{(1)} \cdot a_{(1)} \otimes h_{(2)} \cdot a_{(2)} = (h \cdot a)_{(1)} \otimes (h \cdot a)_{(2)}$ and $\varepsilon(h \cdot a) = \varepsilon(h)\varepsilon(a)$, $h \in H, a \in A$.
- ▶ The canonical map $can : H \otimes C \rightarrow C \otimes C$ given by $can(h \otimes c) = h \cdot c_{(1)} \otimes c_{(2)}$ is a bijection.

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Definition

Let H be a \mathbb{K} -Hopf algebra and C a left H -Hopf Galois co-object. Then $\tau : C \otimes C \rightarrow H$ given by $\tau = (\varepsilon \otimes Id_H) \circ can^{-1}$ is the cotranslation map.

Analogy with theory of groupoids:

The map $can : G_{x,y} \times G_{y,y} \rightarrow G_{x,y} \otimes G_{x,y}$ is a bijection, where $can(g \times h) = (g \times gh)$ for $g \in G_{x,y}$ and $h \in G_{y,y}$. The inverse is given by $can^{-1} : G_{x,y} \times G_{x,y} \rightarrow G_{x,y} \times G_{y,y}$ where $can^{-1}(g \times j) = g \times g^{-1}j$, where $g \in G_{x,y}, j \in G_{x,y}$.

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Definition

For H a Hopf algebra and C a left Galois co-object, the dual Ehresmann-Schauenburg construction $E(C, H)$ by the Hopf algebra defined as

- ▶ $E(C, H) = C \otimes C / I$ as a \mathbb{K} -vector space, where I is the \mathbb{K} -vector subspace of $C \otimes C$ generated by the set

$\{h_{(1)} \cdot c \otimes h_{(2)} \cdot d - \varepsilon(h)c \otimes d; h \in H, c, d \in C\}$. Notation:

$$\sum a_i \otimes b_i + I = \overline{\sum a_i \otimes b_i}.$$

- ▶ The multiplication map is given by

$$\overline{(a \otimes b)(c \otimes d)} = \overline{a \otimes \tau(b \otimes c) \cdot d}$$

and the unit map is given by $1_{E(C, H)} = \overline{u_{(1)} \otimes u_{(2)}}$ where $\varepsilon(u) = 1$.

- ▶ The comultiplication map is given by

$\Delta(\overline{a \otimes b}) = \overline{a_{(1)} \otimes b_{(2)} \otimes a_{(2)} \otimes b_{(1)}}$ for $a, b \in C$. The counit map is given by $\varepsilon(\overline{a \otimes b}) = \varepsilon(a)\varepsilon(b)$ for $a, b \in C$.

- ▶ The antipode map is given by $S(\overline{a \otimes b}) = \overline{\tau(u_{(1)} \otimes a) \cdot b \otimes u_{(2)}}$ where $\varepsilon(u) = 1$.

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Analogy with groups/groupoids:

$$G_{y,y} = G_{y,x} \times G_{x,y} / \sim$$

where \sim is the equivalence relation generated by $cg \times d \sim c \times gd$ for $c \in G_{y,x}, g \in G_{y,y}, d \in G_{y,x}$ with multiplication defined as $(a \times b / \sim)(c \times d / \sim) = ab^{-1}c \times d / \sim = a \times b^{-1}cd / \sim$ and unit any element $a \times a / \sim$ and inverse given by $S(a \times b / \sim) = ca^{-1} \times b$

Definition

Let G, H be Hopf algebras. A $G - H$ -biGalois co-object is a left G -Hopf Galois co-object C that is also a right H -Hopf Galois co-object such that

$$(g \cdot c) \cdot h = g \cdot (c \cdot h)$$

for $g \in G, h \in H$ and $c \in C$.

Fact: For C a right (left) H -Hopf Galois co-object, there is one Hopf algebra G (up to isomorphism) that turns C into a $G - H$ (or $H - G$)-biGalois co-object. This Hopf algebra corresponds to the dual Ehresmann-Schauenburg construction.

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Example of application:

$$\mathbb{K} = \mathbb{Q}$$

$$H_\alpha = \langle 1, c, s, s^2 \rangle$$

(the trigonometric algebra), $C = \mathbb{Q}[\sqrt[4]{2}]$ ($w = \sqrt[4]{2}$) with coalgebra structure given by

$$\Delta(w^i) = \sum_{k+j=i \pmod{4}} \alpha_{(k+j)} w^k \otimes w^j$$

where $\alpha_l = 1, l < 4, \alpha_l = 2, l \geq 4$.

$$s \cdot w^i = \sin(\pi i/2 + \alpha) w^i, c \cdot w^i = \cos(\pi i/2 + \alpha) w^i,$$

$$s^2 \cdot w^i = \sin^2(\pi i/2 + \alpha) w^i, 1 \cdot w^i = w^i$$

C is a biGalois object over all these Hopf algebras

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$$H_\alpha \cong H_\beta, \alpha, \beta \in \{1, \pi/2, \pi, 3\pi/2\}$$

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Theorem

Let \mathcal{H} be a Hopf category. Then

- ▶ $\mathcal{H}_{x,y}$ is a $\mathcal{H}_{x,x} - \mathcal{H}_{y,y}$ -biGalois co-object under the actions $m_{x,x,y}$ and $m_{x,y,y}$.
- ▶ The multiplication map $m_{x,y,z} : \mathcal{H}_{x,y} \otimes \mathcal{H}_{y,z}$ induces an isomorphism of $\mathcal{H}_{x,x} - \mathcal{H}_{z,z}$ biGalois co-objects $\mathcal{H}_{x,y} \otimes_{\mathcal{H}_{y,y}} \mathcal{H}_{y,z} \cong \mathcal{H}_{x,z}$.

Analogy with groupoids: For a (connected) groupoid G , we can say that $G_{x,z} \cong G_{x,y} \times G_{y,z} / \sim$ where \sim is the relation of generated by $gh \times j \sim g \times hj$ for $g \in G_{x,y}, h \in G_{y,y}, j \in G_{y,z}$.

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Concrete Example of Hopf Categories: Harrison Cocycles

Definition

Let H be a Hopf algebra. Then $R = \sum R^1 \otimes R^2 \in H \otimes H$ is a **normalized Harrison cocycle** if there is $R = \sum R^1 \otimes R^2 = \sum r^1 \otimes r^2$ is invertible in $H \otimes H$ (denote its inverse by $\bar{R} = \sum \bar{R}^1 \otimes \bar{R}^2$) and which satisfies the equality

$$\sum R^1 \otimes R_{(1)}^2 r^1 \otimes R_{(2)}^2 r^2 = \sum R_{(1)}^1 r^1 \otimes R_{(2)}^1 r^2 \otimes R^2,$$

$$\sum \varepsilon(R^1) \varepsilon(R^2) = 1,$$

Example of Hopf Categories

Proposition

Let H be a Hopf algebra. Therefore, we have a Hopf category with:

- ▶ We have $X = \{R = \sum R^1 \otimes R^2 \in H \otimes H; R \text{ is a normalized Harrison cocycle of } H\}$,
- ▶ The coalgebra H_{RT} for $R, T \in X$ is given by H as the \mathbb{K} -vector space, with comultiplication given by

$$\Delta_{RT}(h) = \sum R^1 h_{(1)} \bar{T}^1 \otimes R^2 h_{(2)} \bar{T}^2,$$

and counit $\epsilon_{RT}(h) = \epsilon(h)$. for $h \in H$.

- ▶ The multiplication $m_{RTL} : H_{RT} \otimes H_{TL} \rightarrow H_{RL}$ for $R, T, L \in X$ is given by the usual multiplication of H , or

$$m_{RTL}(g \otimes h) = gh,$$

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- ▶ The antipode s_{RT} for $R, T \in H \otimes H$ is given by

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- ▶ The antipode s_{RT} for $R, T \in H \otimes H$ is given by

Example of Hopf Categories

Proposition

Let H be a Hopf algebra. Therefore, we have a Hopf category with:

- ▶ We have $X = \{R = \sum R^1 \otimes R^2 \in H \otimes H; R \text{ is a normalized Harrison cocycle of } H\}$,
- ▶ The coalgebra H_{RT} for $R, T \in X$ is given by H as the \mathbb{K} -vector space, with comultiplication given by

$$\Delta_{RT}(h) = \sum R^1 h_{(1)} \bar{T}^1 \otimes R^2 h_{(2)} \bar{T}^2,$$

and counit $\varepsilon_{RT}(h) = \varepsilon(h)$. for $h \in H$.

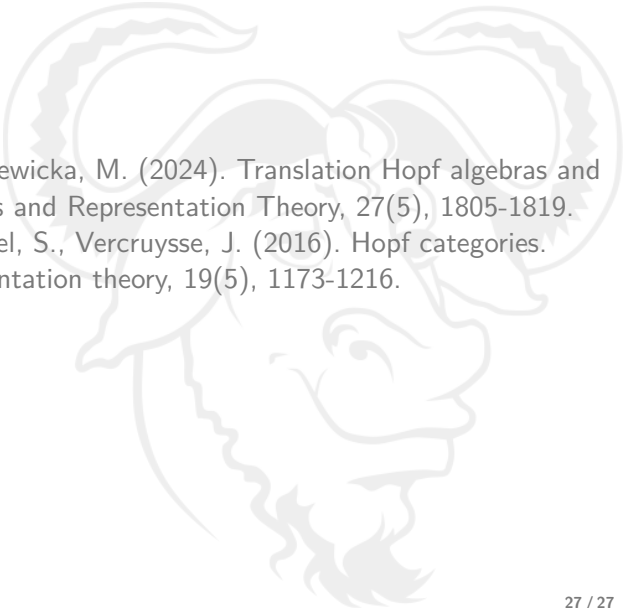


- ▶ The multiplication $m_{RTL} : H_{RT} \otimes H_{TL} \rightarrow H_{RL}$ for $R, T, L \in X$ is given by the usual multiplication of H , or

$$m_{RTL}(g \otimes h) = gh,$$

for $g, h \in H$. The unit $1_{RR} \in H_{RR}$ is given by $1_H \in H$.

- ▶ The antipode s_{RT} for $R, T \in H \otimes H$ is given by

References

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-  Brzeziński, T., Hryniewicka, M. (2024). Translation Hopf algebras and Hopf heaps. *Algebras and Representation Theory*, 27(5), 1805-1819.
-  Batista, E., Caenepeel, S., Vercruyssen, J. (2016). Hopf categories. *Algebras and representation theory*, 19(5), 1173-1216.