Geometries admitting trialities

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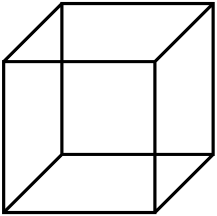
Incidence Geometry

A geometry Γ consists of

- A set of objects X.
- A type function $t\colon X \mapsto I$.
- A reflexive and symmetric incidence relation $\star$ such that if $x \star y$ and $t(x) = t(y)$ then $x = y$.
- A subset $F \subset X$ of two by two incident elements is called a *flag*. A flag has to be a subset of a flag of maximal size (called a *chamber*).

For the cube :

- X consists of 8 points, 12 lines and 6 faces.
- $I = \{point, line, face\}$. We say that the geometry has rank 3.
- The relation $\star$ is symmetric inclusion.



A geometry $\Gamma(X, \star, t)$ consists of

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- A type function $t \colon X \mapsto I$.
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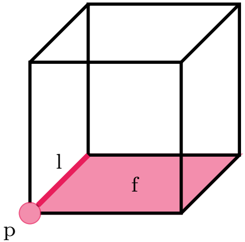
- The relation $\star$ is symmetric inclusion.
- The set $\{p, l, f\} \subset X$ is a flag.



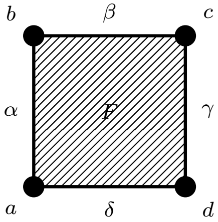
Incidence graph

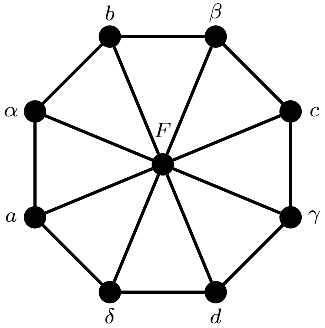
Given a geometry Γ, one can construct its associated incidence graph.

- For each object $x \in X$, create a vertex and label it by x.
- There is an edge between two vertices labelled x and y if and only if $x \star y$ in Γ.

.. and its incidence graph.

A square





It is 3-partite because the square has 3 types of elements.

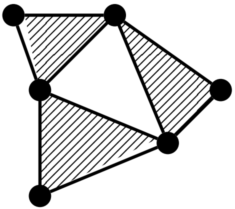
With no restrictions, the realm of possible geometries is huge. Symmetry is nice so let's ask for some.

Let Γ be a geometry. $\mathrm{Aut}(\Gamma)$ is the group of incidence preserving and type preserving bijections from X to X.

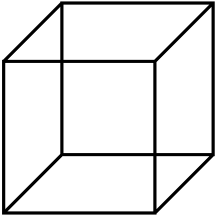
Γ is Flag Transitive $\iff$ $\mathrm{Aut}(\Gamma)$ acts transitively on the flags of the same type.

Γ is Residually Connected $\iff$ Incidence graph is connected and the neighbours of an element is a connected subgraph.

An example not FT and not RC.

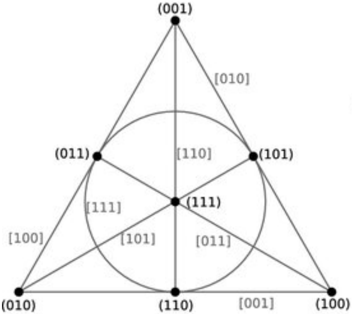


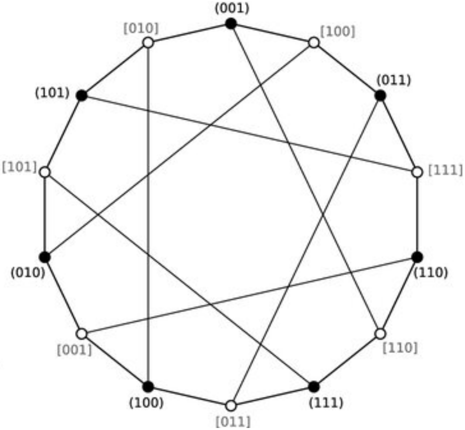
The cube is FT and RC.



Duality in the projective plane

The *Projective plane* constructed over the field with 2 elements (also called Fano Plane) $PG(2,2)$ on the left and its incidence graph on the right. It has 7 points and 7 lines. Two lines intersect in exactly one point and two points share one line with whom they are incident with.

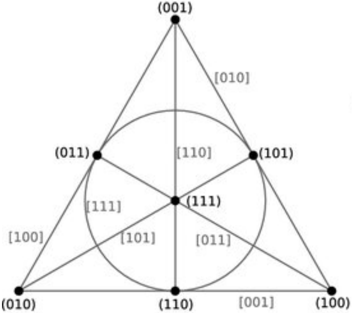


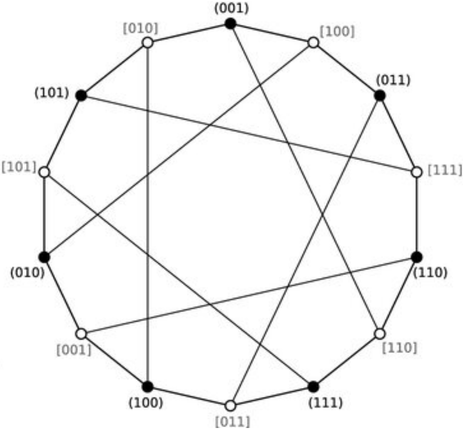


Credits : Martin Mačaj and Primoz Sparl, Half-Arc-Transitive Graphs and the Fano Plane, Graphs and Combinatorics 37(3), 2021.

Duality in the projective plane

$\mathrm{Cor}(\Gamma)$ is the group of incidence preserving bijections from X to X but can change the types under the condition that for $\sigma \in \mathrm{Cor}(\Gamma)$, if $t(x) = t(y)$ then $t(\sigma(x)) = t(\sigma(y))$. A *duality* is an element of order 2 in $\mathrm{Cor}(\Gamma)$ that exchanges two types.





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What about trialities?

For a rank 3 geometry, an element of $\mathrm{Cor}(\Gamma)$ that cyclically permutes the types and is of order 3 is called a *triality*.
We can now ask the following question :

> Does there exist rank 3 flag transitive, residually connected geometries admitting trialities and no dualities?

Yes ! .. But the only examples so far used heavy group theoretical machinery to get produced.

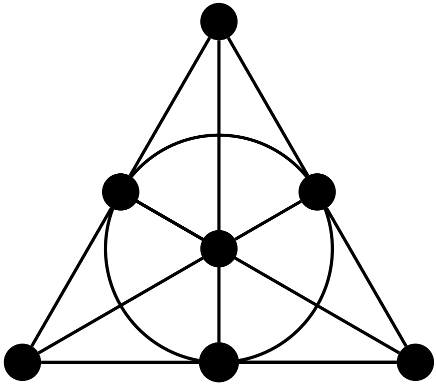
The triangle complex

A prototype for trialities : the triangle complex. Let $\Gamma = (\mathcal{P} \sqcup \mathcal{L}, \star, t)$ be a rank two geometry. The *triangle complex* $\Delta(\Gamma) = (X_{\Delta(\Gamma)}, \star_{\Delta(\Gamma)}, t_{\Delta(\Gamma)})$ over $I = \{1, 2, 3\}$ is the rank three incidence system constructed from Γ in the following way:

1. The set $X_{\Delta(\Gamma)}$ of elements of $\Delta(\Gamma)$ is the set of all the triples (p, L, i) with $p \in \mathcal{P}$, $L \in \mathcal{L}$, $i \in \{1, 2, 3\}$ satisfying $p \star L$.
2. The incidence relation $\star_{\Delta(\Gamma)}$ is defined by $(p, L, i) \star_{\Delta(\Gamma)} (p', L', i \bmod 3 + 1)$ if and only if $p \star L'$ and $p' \not\star L$.
3. The type function $t_{\Delta(\Gamma)} : X_{\Delta(\Gamma)} \mapsto \{1, 2, 3\}$ is defined by $t_{\Delta(\Gamma)}((p, L, i)) = i$.

To draw them we will use the colorcode $1 =$ red, $2 =$ green and $3 =$ blue.

$\Gamma = PG(2, 2)$.



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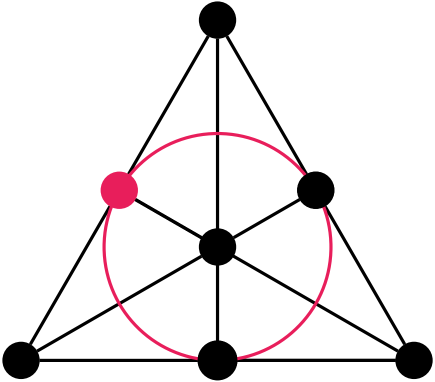

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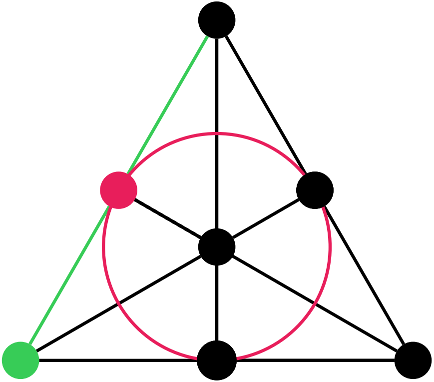

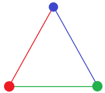

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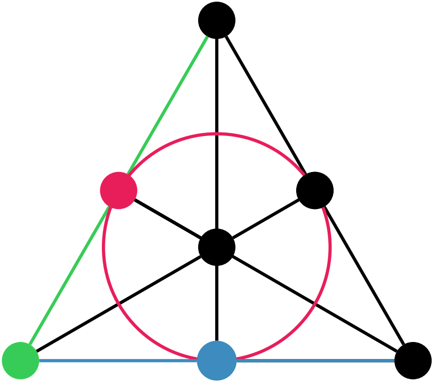
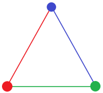


The chambers of $\Delta(\Gamma)$ look like triangles hence the name "triangle complex". If Γ is a linear spaces (by two points passes exactly one line and there is at least 2 lines), $\Delta(\Gamma)$ is always a geometry.

The triangle complex



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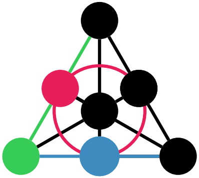
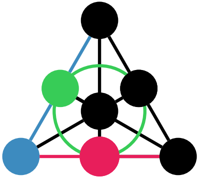
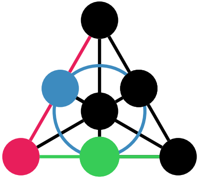
3. The type function $t_{\Delta(\Gamma)} : X_{\Delta(\Gamma)} \mapsto \{1, 2, 3\}$ is defined by $t_{\Delta(\Gamma)}((p, L, i)) = i$.
4. Moreover, the triangle complex admits trialities by design. One of them is $\tau(p, L, i) = (p, L, i \bmod 3 + 1)$ which corresponds to cyclically changing the colors while keeping the same points and lines.
   red $\mapsto$ green $\mapsto$ blue $\mapsto$ red.







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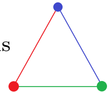
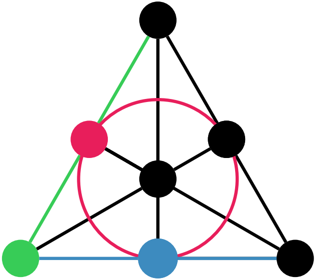
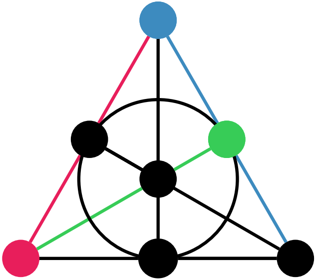
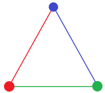


It is flag transitive because the automorphism group of the projective plane is transitive on ordered triples of non collinear points (trust me).

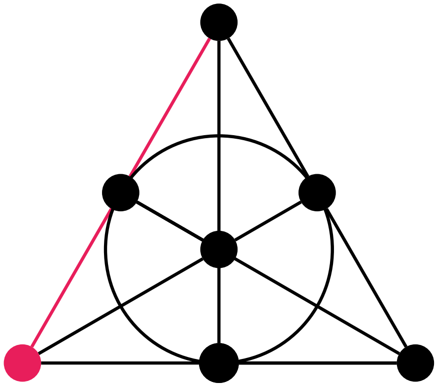
- Let $\{(p_1, L_1, 1), (p_2, L_2, 2), (p_3, L_3, 3)\}$ the chamber on the left.
- Let $\{(p'_1, L'_1, 1), (p'_2, L'_2, 2), (p'_3, L'_3, 3)\}$ the chamber on the right.
- Pick $\theta \in \mathrm{Aut}(PG(2,2))$ such that $\theta(p_1) = p'_1, \theta(p_2) = p'_2, \theta(p_3) = p'_3$
- Then $\nu(p, L, i) = (\theta(p), \theta(L), i)$ sends the chamber on the left to the one on the right.

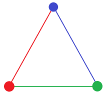






1. It is flag transitive.
2. It is connected. Is it residually connected?





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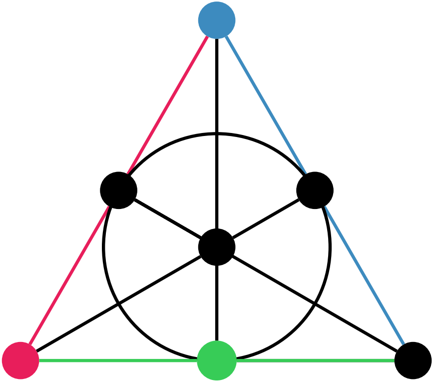
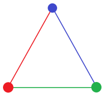
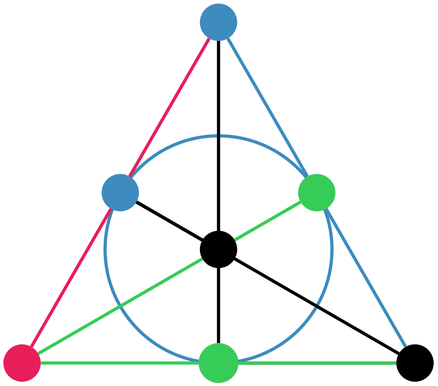


The triangle complex $\Delta(PG(2,2))$

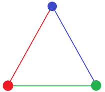


1. It is flag transitive because the automorphism group of the projective plane is transitive on ordered triples of non collinear points (trust me).
2. It is connected. Is it residually connected? Yes !



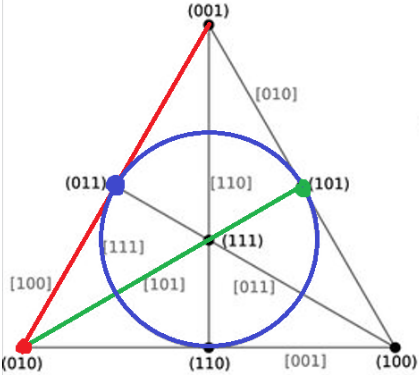
Dualities?

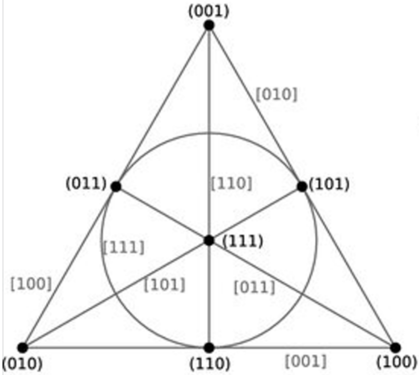
Unfortunately, $\Delta(\Gamma)$ inherits dualities from the projective plane. Indeed, if α is a duality of Γ, then



$$\beta(p, L, i) = \begin{cases} (\alpha(L), \alpha(p), 1), & \text{if } i = 1 \\ (\alpha(L), \alpha(p), 3), & \text{if } i = 2 \\ (\alpha(L), \alpha(p), 2), & \text{if } i = 3 \end{cases}$$

is a duality of $\Delta(\Gamma)$.





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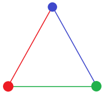

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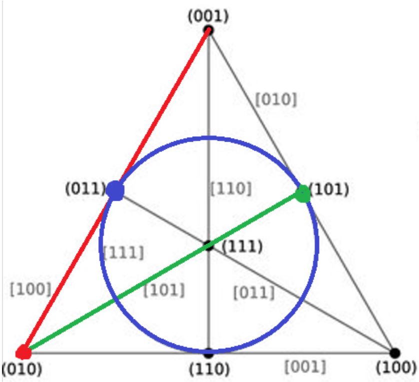

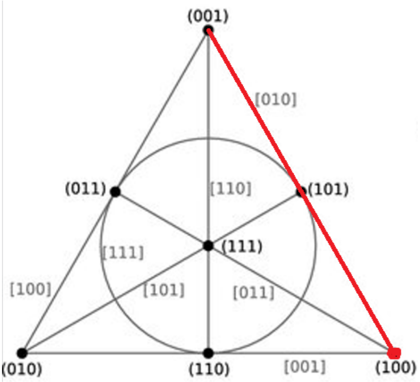

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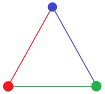

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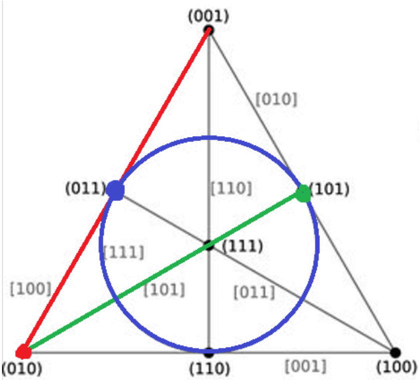

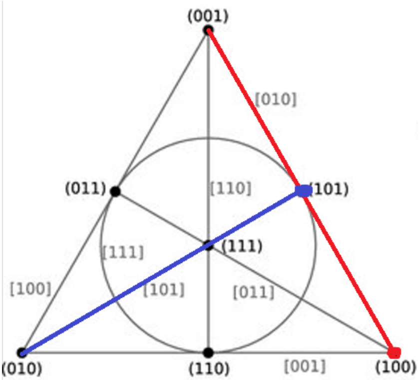

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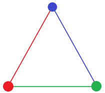

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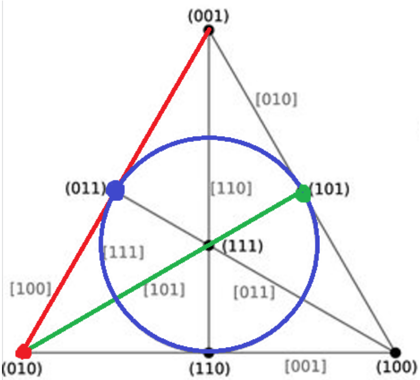
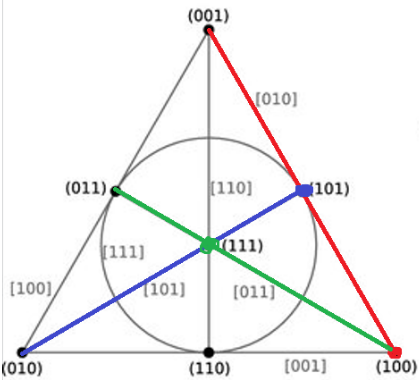
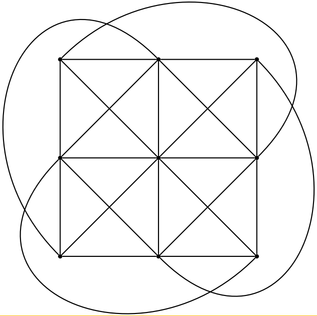




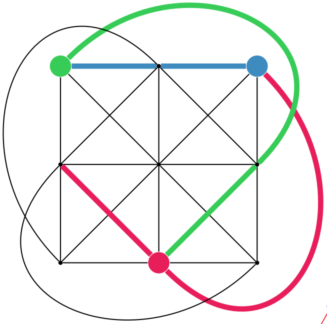
All hope is lost?

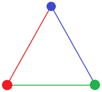
Is all hope lost? Not quite, we can do the same construction over Γ an affine plane. The affine plane is very symmetric and does not admit dualities so there is hope !

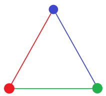
This affine plane $AG(2,3)$ has 9 points and 12 lines. Every line is incident to 3 points, every point is incident to 4 lines. It has the property that by two points passes exactly one line.



A chamber in $\Delta(\Gamma)$ when $\Gamma = AG(2,3)$.







1. $\Delta(\Gamma)$ is flag transitive because $\mathrm{Aut}(AG(2,3))$ is transitive on ordered triples of non collinear points.
2. $\Delta(\Gamma)$ is residually connected.
3. $\Delta(\Gamma)$ admits trialities by design (cyclically changes the colors).
4. Does it admit dualities?

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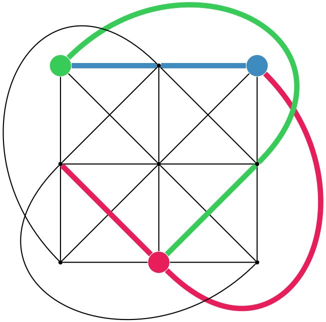
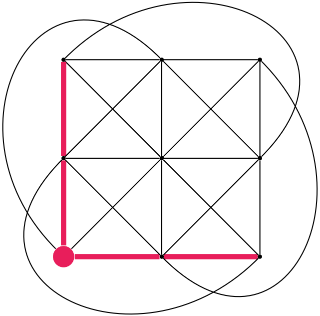
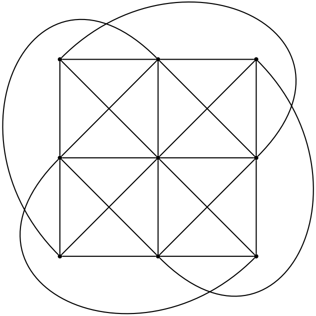


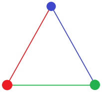
$\Delta(\Gamma)$ does not admit dualities when Γ is an affine plane. Let's prove it by a counting argument. Assume there is a duality β that fixes elements of type 1 and exchanges elements of type 2 and 3.

Two elements of type 1 in $\Delta(\Gamma)$. How many elements of type 2 are incident to both of them?



After the action of β, ..

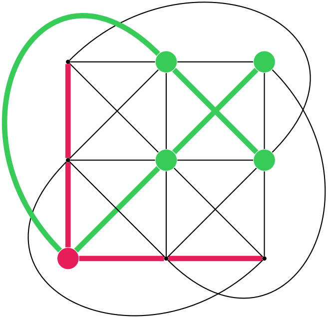




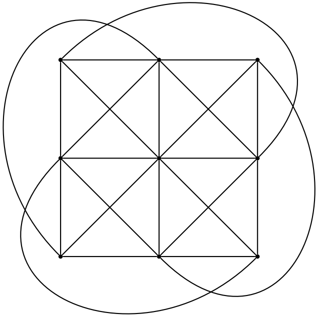
The triangle complex $AG(2,3)$

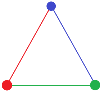
$\Delta(\Gamma)$ does not admit dualities when $\Gamma = AG(2,3)$. Let's prove it by a counting argument. Assume there is a duality β that fixes elements of type 1 and exchanges elements of type 2 and 3.

Two elements of type 1 in $\Delta(\Gamma)$. They have 4 elements of type 2 in common with whom they are incident.



After the action of β, ..

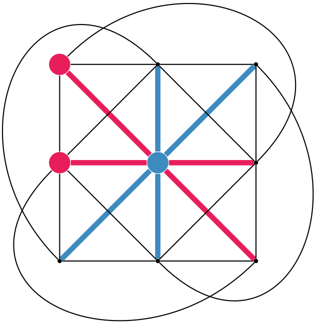


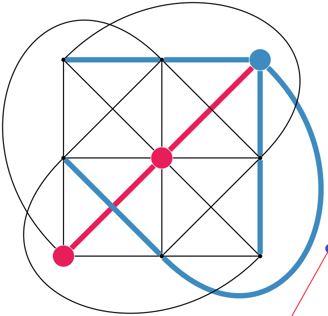


After the action of β, if both lines of type 1 don't intersect, there are no elements of type 3 incident to both of them. In any case, the number of type 3 incident to both the image of the elements of type 1 is either 0,2 or 3 but never 4. A contradiction.

After the action of β, if the two elements of type 1 have lines intersecting in one point, there are 2 elements of type 3 incident to both of them.

After the action of β, if both lines of elements of type 1 are sent to the same line, there are 3 type 3 elements incident to both of them.





- The geometry built over any affine plane $AG(2,q), q \geq 3$ is residually connected, flag transitive, admits trialities and no dualities
- The geometry built over any projective plane $PG(2,q), q \geq 2$ is residually connected, flag transitive, admits trialities and dualities.
- Moreover, one can prove that all automorphisms of $\Delta(\Gamma)$ in these cases are induced by automorphisms of Γ. Namely if $\theta \in \mathrm{Aut}(\Delta(\Gamma))$ then there exists $\mu \in \mathrm{Aut}(\Gamma)$ such that $\theta((p,L,i)) = (\mu(p), \mu(L), i)$.
- The geometry we have seen on the affine plane $AG(2,3)$ is the smallest one we know so far (FT, RC, trialities no dualities). It has 108 elements, 36 of each type and 432 chambers.

Thank you ! If you have any questions feel free to ask !